Transpose : Linear Algebra Notes

Satya Mandal

September 22, 2005

Let $\mathbb{F}$ be a filed and V be vector space over F with $\dim(V) = n < \infty$.

Definition 0.1 *Let V, W be two vector spaces over $\mathbb{F}$ and*

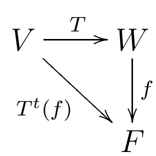
$$T : V \to W$$

be a linear trancformation. We define a map

$$T^t : W^* \to V^*$$

by defining $T^t(f) = foT : V \to \mathbb{F}$ for $f \in W^*$.

Diagramatically:



We say that T^t is the transpose of T. Note that the transpose T^t is linear transformation.

Theorem 0.1 *Let V, W be two finite dimensional vector spaces over $\mathbb{F}$, with $\dim(V) = n$ and $\dim(W) = m$. Let*

$$T : V \to W$$

be a linear transformation. Assume $e_1, \ldots, e^n$ is a basis of V and $\epsilon_1, \ldots, \epsilon_m$ is a basis of W. Suppose A is the matrix of T with respect to these two bases. Then the transpose A^t is the matrix of T^t with respect to the dual bases $\epsilon_1^*, \ldots, \epsilon_m^*$ of W^* and $e_1^*, \ldots, e_n^*$ of V^*.

Proof. We have

$$(T(e_1), \dots, T(e_n)) = (\epsilon_1, \dots, \epsilon_m)A.$$

Write $A = (a_{ij})$. Apply, for example, ϵ_1^* to the above equation, we get

$$(\epsilon_1^*(T(e_1)), \dots, \epsilon_1^*(T(e_n))) = (1, 0, \dots, 0)A = (a_{11}, \dots, a_{1n}).$$

This means that

$$T^t(\epsilon_1^*) = (e_1^*, \dots, e_n^*) \begin{pmatrix} a_{11} \\ a_{12} \\ \cdots \\ a_{1n} \end{pmatrix}.$$

Similarly, for $i = 1, \dots, m$, we work with ϵ_i^*, and get

$$T^t(\epsilon_i^*) = (e_1^*, \dots, e_n^*) \begin{pmatrix} a_{i1} \\ a_{i2} \\ \cdots \\ a_{in} \end{pmatrix}.$$

Therefore,

$$(T^t(\epsilon_1^*), \dots, T^t(\epsilon_m^*)) = (e_1^*, \dots, e_n^*)A^t.$$

Hence A^t is the matrix of T^t, with respect to these dual bases. So, the proof is complete.

Theorem 0.2 *Let V, W be two finite dimensional vector spaces over $\mathbb{F}$, with $\dim(V) = n$ and $\dim(W) = m$. Let*

$$T : V \to W$$

be a linear transformation.

1. *Let $\mathcal{R} = T(V)$ be the range of T and $\mathcal{N}$ be the null space of the transpose T^t. Then the null space*

$$\mathcal{N} = ann(\mathcal{R})$$

2. *$rank(T) = rank(T^t)$.*

3. *Also $range(T^t) = ann(N_T)$, where N_T is the null space of T.*

Proof. To prove (1), we have null space of $T^t = \mathcal{N} = \{f \in W^* : T^t(f) = 0\} = \{f \in W^* : foT = 0\} = \{f \in W^* : f(\mathcal{R}) = 0\} = ann(\mathcal{R})$.

To prove (2), note that

$$\dim(W) = \dim(\mathcal{R}) + \dim(ann\mathcal{R}) = rank(T) + \dim(\mathcal{N})$$

and

$$\dim(W) = \dim(W^*) = rank(T^t) + \dim(\mathcal{N}).$$

Therefore $rank(T) = rank(T^t)$ and the proof of (2) is complete.

Proof of (3) is similar to that of (1).

Theorem 0.3 *Let A be an $m \times n$ matrix with entries in $\mathbb{F}$. Then $Row - rank(A) = Column - rank(A)$.*

Proof. Let $T : \mathbb{F}^n \to \mathbb{F}^m$ be the linear transformation given by $T(X) = AX$. Observe that $T(\mathbb{F}^n) =$ the row space of A.

By above theorem $\dim(T(X)) = rank(T) = rank(T^t)$. So, $Row-rank(A) = rank(T) = rank(T^t) = Row - rank(A^t) = column - rank(A)$. Therefore the proof is complete.

Remark 0.1 *One should try a direct proof of this theorem.*