K-theory, Chow groups and Riemann-Roch

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October 26, 2004

We will work over a quasi-projective variety over a field , though many statements will work for arbitrary Noetherian schemes. The basic reference is [2]. Facts about K^0, K_0 can be found in [4]. Of course most of this was originally concieved by Grothendieck (and a few others) in this generality and their accounts can be found in [3, 1].

1 K^0 and K_0 of schemes

Let X be a quasi-projective variety. We define $K^0(X)$ as follows. Let $F_{vb}(X)$ be the free abelian group on all isomorphism classes of vector bundles (of finite rank) over X and consider the equivalence relation $R_{vb}(X)$ generated by the following. If $0 \to E_1 \to E_2 \to E_3 \to 0$ is an exact sequence of vector bundles, then $[E_2] = [E_1] + [E_3]$. The quotient $F_{vb}(X)/R_{vb}(X)$ is called $K^0(X)$. Similarly we define $K_0(X)$ by taking the free abelian group $F_{coh}(X)$ generated by all isomorphism classes of coherent sheaves on X modulo the relation $R_{coh}(X)$ generated by $[\mathcal{F}_2] = [\mathcal{F}_1] + [\mathcal{F}_3]$ whenever we have an exact sequence of coherent sheaves, $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$. $K^0(X)$ is a commutative ring where multiplication is given by tensor products and $K_0(X)$ is a module over $K^0(X)$, the action given by tensor product. We also have a natural map $K^0(X) \to K_0(X)$ given by $[E]$ going to itself. We may also consider the free abelian group on all isomorphism classes of coherent sheaves of *finite projective dimension* and going modulo a relation exactly as above and let me temporarily call this group $L(X)$. We have natural homomorphisms, $K^0(X) \to L(X) \to K_0(X)$. Easy to check that the first map is an isomorphism. We will see that all are isomorphisms if X is smooth. K^0 is a

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functor from the category of schemes to category of commutative rings, which is contravariant. If $X = \operatorname{Spec} R$ for a ring R, we may write $K_0(R), K^0(R)$ instead of $K_0(X), K^0(X)$.

We start with some elementary facts about these groups. Any lemma with a † should be considered routine and the reader is urged to work out a proof before reading the proof.

Exercises:

1. Show that for any irreducible variety X, there exists a natural homomorphism $\mathrm{rk} : K^0(X) \to \mathbb{Z}$, where $\mathrm{rk}(E)$ is the rank of the vector bundle E.
2. Compute K_0, K^0 (which should be equal from what we mentioned before) for $X = \mathbb{A}^1, \mathbb{P}^1$.
3. Show that for any curve X, $K^0(X) = \mathbb{Z} \oplus \operatorname{Pic} X$, where $\mathbb{Z}$ corresponds to rank as above.
4. Show that $K^0(X) = \mathbb{Z}$, where $X = \operatorname{Spec} R$, R a local (Noetherian) ring.
5. Let $A = \mathbb{C}[\![t^2, t^3]\!]$ and $R = A[T]$. Show that $K^0(R) \neq \mathbb{Z}$. Can you compute it?

Next, we list some properties of these groups.

Property 1.1 (Functoriality) *If $f : X \to Y$ is any morphism, we have an induced map of rings, $f^* : K^0(Y) \to K^0(X)$, given by $f^*[E] = [f^*E]$, where f^*E is just the pull back of the vector bundle.*

Proof: We certainly get a map f^* from $F_{vb}(Y)$ to $F_{vb}(X)$ defined by, $f^*[E] = [f^*E]$. If $0 \to E_1 \to E_2 \to E_3 \to 0$ is an exact sequence of vector bundles on Y, then $0 \to f^*E_1 \to f^*E_2 \to f^*E_3 \to 0$ is exact and thus we get an induced map $f^* : K^0(Y) \to K^0(X)$ as claimed. □

Property 1.2 (Flat pull-back) *If $f : X \to Y$ is a flat morphism, then we have a (flat) pull back, $f^* : K_0(Y) \to K_0(X)$ defined exactly as above.*

Proof: The proof is exactly the same as above since given an exact sequence of sheaves $0 \to F_1 \to F_2 \to F_3 \to 0$ on Y, the induced complex, $0 \to f^*F_1 \to f^*F_2 \to f^*F_3 \to 0$ is exact, since f is flat. □

Property 1.3 (Proper push-forward) *If $f : X \to Y$ is a proper morphism, then we have a map $f_* : K_0(X) \to K_0(Y)$ defined as, $f_*([\mathcal{F}]) = \sum(-1)^i[R^i f_*\mathcal{F}]$.*

Proof: This follows from the fact that given an exact sequence, $0 \to F_1 \to F_2 \to F_3 \to 0$ on X, we have a long exact sequence,

$$0 \to f_*F_1 \to f_*F_2 \to f_*F_3 \to R^1 f_*F_1 \to R^1 f_*F_2 \to \cdots$$

$$\to R^n f_*F_2 \to R^n f_*F_3 \to 0,$$

where $n = \max\{\dim f^{-1}(y) \mid y \in Y\}$. □

Property 1.4 (Projection Formula) *If $f : X \to Y$ is a proper morphism and if $\alpha \in K_0(X)$ and $\beta \in K^0(Y)$ then $f_*(f^*(\beta) \cdot \alpha) = \beta \cdot f_*(\alpha)$.*

Proof: Clearly, suffices to prove this for $\alpha = F$, a coherent sheaf on X and $\beta = E$, a vector bundle on Y. This follows from the usual projection formula,

$$R^i f_*(f^*E \otimes F) = E \otimes R^i f_*F,$$

for all i. □

Lemma 1 *The natural map $K_0(X_{\mathrm{red}}) \to K_0(X)$ is an isomorphism.*

Proof: Let I be the sheaf of ideals in $\mathcal{O}_X$ defining $Z = X_{\mathrm{red}} \subset X$. Then any coherent sheaf F on X such that $IF = 0$ is in fact a sheaf over Z. Since $I^n = 0$ for some n, we may filter any coherent sheaf F on X as,

$$0 = I^n F \subset I^{n-1}F \subset \cdots \subset IF \subset F.$$

Then, by our relation, we see that $[F] = \sum[I^k F/I^{k+1}F]$ in $K_0(X)$. Since each of the terms come from Z, we see that $K_0(Z) \to K_0(X)$ is onto. (Notice that there are no higher direct images in this situation.)

To check that this map is an isomorphism, we construct an inverse to the above map. By an elementary induction on n, we may assume that $I^2 = 0$. We define a map $f : F_{vb}(X) \to K_0(Z)$ as follows. If F is any coherent sheaf on X define $f(F) = [IF] + [F/IF]$. Notice that both $IF, F/IF$ are shevaes on Z. We claim that if $G \subset F$ with $IG = I(F/G) = 0$, then $f(F) = [G] + [F/G] \in K_0(Z)$. Since we have a surjective map $F \to F/G$ and

$I(F/G) = 0$, we get an induced surjective map $p : F/IF \to F/G$. Thus we get a commutative diagram,

$$\begin{array}{ccccccccc} 0 & \to & IF & \to & F & \to & F/IF & \to & 0 \\ & & \downarrow q & & \| & & \downarrow p & & \\ 0 & \to & G & \to & F & \to & F/G & \to & 0 \end{array}$$

Thus we get that $\ker p \cong \operatorname{coker} q$. So,

$$f(F) = [IF] + [F/IF] = [G] - [\operatorname{coker} q] + [\ker p] + [F/G] = [G] + [F/G]$$

Next we check that f respects our relation in $K_0(X)$. If $0 \to F \to G \to H \to 0$ is an exact sequence of coherent sheaves let $H_1 = \mathrm{im}, IG$ and let $H_2 = H/H_1$. Similarly, let $F_1 = IG \cap F$ and $F_2 = F/F_1$. Then we have a commutative diagram,

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F_1 & \to & IG & \to & H_1 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F & \to & G & \to & H & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & F_2 & \to & G/IG & \to & H_2 & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Since all the sheaves occurring above, except the ones in middle row are sheaves on Z, we get,

$$\begin{aligned} f([G] - [F] - [H]) &= ([G/IG] + [IG]) - ([F_2] + [F_1]) - ([G_2] + [G_1]) \\ &= ([G/IG] - [H_2] - [F_2]) + ([IG] - [H_1] - [F_1]) = 0. \end{aligned}$$

This finishes the proof, since it is clear that the f constructed is indeed an inverse to the surjective map $K_0(Z) \to K_0(X)$. □

Lemma 2 *If $X = X_1 \cup X_2 \cup \cdots X_n$, where X_i's are connected componenets of X, then $K_0(X) = K_0(X_1) \oplus K_0(X_2) \cdots \oplus K_0(X_n)$. Similarly for K^0.* †

Proof: We will assume that $X = X_1 \cup X_2$, the rest being clear. We will prove the statement for K_0, the case for K^0 being identical. If F is a coherent sheaf on X, it clear that it is just $F_{|X_1} \oplus F_{|X_2}$ and the rest is obvious. □

1.1 Extending sheaves

In this section, X will be a Noetherian scheme, $Z \subset X$ a closed subscheme and $U = X - Z$.

Lemma 3 *Let $X = \operatorname{Spec} R$, an affine Noetherian scheme, Z, U as above. If F is any coherent sheaf on U, then it is globally generated.*

Proof: Let $I \subset R$ be the ideal defining Z and let $f_1, \ldots, f_r \in I$ be a set of generators of I. Let $U_i \subset U$ be the affine open susbet $f_i \neq 0$. Then U_i's cover U. If F is a coherent sheaf on U, its restriction to U_i is coherent and since U_i is affine, this restriction is given by a finitely generated module M_i over R_{f_i}. If $m \in M_i$, then easy to see that $f_i^n m$ is a global section of F for $n >> 0$. Now, picking generators $m_{ij} \in M_i$, we can choose integers n_{ij} so that $s_{ij} = f_i^{n_{ij}} m_{ij}$ are global sections of F. Immediate that these s_{ij} generate F.

$\square$

Lemma 4 *If G is a coherent sheaf on X and $H' \subset G_{|U}$ is a subsheaf, then there exists a subsheaf $H \subset G$ so that $H_{|U} = H'$.*

Proof: We define a presubsheaf $H'' \subset G$ as follows. For any open set $V \subset X$, define $H''(V)$ to be the inverse image of $H'(U \cap V) \subset G(U \cap V) = G_{|U}(V)$ under the natural restriction map $G(V) \to G(U \cap V)$. Define H to be the associated sheaf. One easily checks that H has all the required properties. $\square$

Lemma 5 *If G is a coherent sheaf on U, there exists a coherent sheaf F on X such that $F_{|U} \cong G$.*

Proof: By Noetherian induction, suffices to show that, if $U \subset V \subset X$ is the largest open set to which G extends (as a coherent sheaf) then $V = X$. So, we may as well start with the situation $U = V$ and if $V \neq X$, then extend G to a strictly larger open set. So, pick a point in Z (which is assumed to be non-empty) and let V be an affine open neighbourhood. We will show that G can be extended to $U \cup V$, proving the lemma. To show this, clearly, it suffices to show that we can extend the coherent sheaf $G_{|U \cap V}$ to a coherent sheaf on V. Thus we are reduced to proving that if V is affine and $W \subset V$ an open subset with a coherent sheaf G on W, then we can extend this to V as a coherent sheaf.

By lemma [3], G is globally generated. So, we have $\mathcal{O}_W^n \to G \to 0$ for some n. Let K be the kernel. Since $\mathcal{O}_W = \mathcal{O}_{V|W}$, by lemma [4], we have $K' \subset \mathcal{O}_V^n$ such that $K'_{|W} = K$. Now it is clear that if we define $F = \mathcal{O}_V^n/K'$ then $F_{|U} \cong G$.

$\square$

Property 1.5 (Localization) *If $Z \subset X$ is a closed subscheme and $U = X - Z$, then we have an exact sequence,*

$$K_0(Z) \to K_0(X) \to K_0(U) \to 0,$$

where the first map is just $[\mathcal{F}]$ going to itself (a coherent sheaf on Z is naturally a coherent sheaf on X, and coincides with the proper push-forward, since clsoed immersions are proper) and the second map is just restriction (which is just the flat pull back, since open immersions are flat).

Proof: It is clear that we have a complex as described above, since the composite map from $K_0(Z) \to K_0(U)$ is zero. Also from the previous lemma, [5], we see that the last map is onto. So, we need only prove exactness at the middle.

We do this by exhibiting a map from $K_0(U) \to P = K_0(X)/K_0(Z)$ which is an inverse to the map given as above. So, we attempt to define a map $\theta : F_{coh}(U) \to P$ by $G \mapsto [F]$ where G is any coherent sheaf on U and F any extension to X assured by lemma [5]. I claim that this map is well defined. That is, the map does not depend on the choice of F. So, let F, F' be two extensions. Then we have

$$G \subset (F \oplus F')_{|U} = G \oplus G,$$

sitting as the diagonal. So, by lemma [4], we can find a $H \subset F \oplus F'$ such that $H_{|U} = G$ sitting diagonally. Considering the projection, $H \to F$, we see that this map is an isomorphism on U. So, the kernel and cokernel are sheaves supported on Z and thus $[H] = [F] \in P$. Similarly, we see that $[H] = [F']$ and thus $[F] = [F'] \in P$. Thus θ is well defined.

Next let $0 \to F \to G \to H \to 0$ be any exact sequence of coherent sheaves on U. Let G' be an extension of G to X. Then by lemma [4], we can find $F' \subset G'$ such that this restricts to $F \subset G$ on U. Let H' be the cokernel and then we have $0 \to F' \to G' \to H' \to 0$, exact on X, which restricts to the original exact sequence on U. But, by the previous pargraph, we know that

$$\theta([G] - [F] - [H]) = [G'] - [F'] - [H'] = 0 \in P.$$

Thus we get a map $\theta : K_0(U) \to P$. The rest is clear. $\square$

Property 1.6 *Let $p : X \times \mathbb{A}^1 \to X$ be the projection, which is flat. Then the natural map $p^* : K_0(X) \to K_0(X \times \mathbb{A}^1)$ is an isomorphism. More generally, let E be a vector bundle on X and consider the natural map $p : E \to X$, where we have called by the same letter E, the associated scheme, then the natural map p^* is an isomorphism on K_0.*

Proof: First, we show that the map p^* is onto. Proof is by induction on $\dim X$. If $\dim X = 0$, then by lemmas [1, 2], we may assume that $X = \operatorname{Spec} k$. Then a coherent sheaf on $X \times \mathbb{A}^1$ is just a finiteley generated module over $k[T]$. Any finitely generated module M has a resolution,

$$0 \to F_1 \to F_0 \to M \to 0$$

over $k[T]$, where F_i's are finitely genertaed free modules and thus $[M] = [F_0] - [F_1]$, which is just the class of a free module of rank $\text{rank}\,(F_0) - \text{rank}\,(F_1)$ and this class clearly comes from $K_0(k)$.

Now, X be arbitrary and property proved for smaller dimensions. Again, by lemma [1], we may assume that X is reduced, since $(X \times \mathbb{A}^1)_{red} = X_{red} \times \mathbb{A}^1$. If X_i's are the irreducible components of X, let $Z = \cup_{i \neq j}(X_i \cap X_j)$. Then Z is closed in X, $\dim Z < \dim X$ and thus by localisation [property 1.5], suffices to prove the surjectivity of $K_0(X - Z) \to K_0((X - Z) \times \mathbb{A}^1)$. But $X - Z$ is the disjoint union of its connected components and by lemma [2], we may assume that X is irreducible (and reduced). Let $U \subset X$ be an affine open set and $Z = X - U$. Then $\dim Z < \dim X$ and so by localisation [property 1.5], we may assume that X is affine. So, let $X = \operatorname{Spec} R$, where R is an integral domain. Let M be any finitely generated module. Let, $0 \to G \to F \to M \to 0$ be a resolution, where F is free. Then G is torsion free and thus $G \otimes_R K$, where K is the fraction field of R, is a free module over $K[T]$. Thus, we may find an $0 \neq f \in R$ such that $G_f \cong R_f[T]^m$. Again, by localisation and induction, we may replace R by R_f and then the class of M is just the difference of two free modules, which comes from R. This proves the result.

By induction on n, we easily see that $K_0(X) \to K_0(X \times \mathbb{A}^n)$ is onto. If E is a vector bundle, then we may find an open set $U \subset X$ such that, if $Z = X - U$, $\dim Z < \dim X$ and $E_{|U} = U \times \mathbb{A}^n$, where $n = \operatorname{rank} E$. Now, using localization and induction on $\dim X$, we see that $p^* : K_0(X) \to K_0(E)$ is onto.

Finally, to show this map is an isomorphism, we proceed as follows. Let $Y \subset E$ be the zero section. Then we have $p : Y \to X$ an isomorphism. Also, we have a resolution of $\mathcal{O}_Y$,

$$0 \to \wedge^n E \to \wedge^{n-1} E \to \cdots \wedge^2 E \to E \to \mathcal{O}_E \to \mathcal{O}_Y \to 0.$$

We define a map $F_{coh}(E) \to K_0(Y) = K_0(X)$ by,

$$F \mapsto \sum (-1)^i [\mathrm{Tor}^i_{\mathcal{O}_E}(F, \mathcal{O}_Y)].$$

Using the resolution above, we see that this is a finite sum and this map factors through $K_0(E)$. If G is a coherent sheaf over X, then $\mathrm{Tor}^i_{\mathcal{O}_E}(p^*G, \mathcal{O}_Y) = 0$ for $i > 0$ and

$$\mathrm{Tor}^0_{\mathcal{O}_E}(p^*G, \mathcal{O}_Y) = p^*G \otimes_{\mathcal{O}_E} \mathcal{O}_Y = G,$$

with the identification of X with Y via p. Thus the map we defined above from $K_0(E) \to K_0(X)$ is just the inverse of the surjective map p^*, proving injectivity of p^*. $\square$

Property 1.7 *Let E be a vector bundle of rank r on X and let $p : Y = \mathbb{P}(E) \to X$ be the natural morphism. Let ξ be the class of the tautological bundle $\mathcal{O}(1)$ in $K^0(Y)$. Then any element in $K_0(Y)$ can be represented uniquely as $\sum_{i=0}^{r-1} p^*(a_i)\xi^i$ for $a_i \in K_0(X)$.*

Proof: First we show that there is a relation,

$$\xi^r + p^*(a_1)\xi^{r-1} + \cdots + p^*(a_r) = 0 \in K_0(Y).$$

We have a natural surjective map, $p^*E \to \xi$ which gives a long exact sequence,

$$0 \to p^*(\wedge^r E)(-r) \to p^*(\wedge^{r-1} E)(-r+1) \to \cdots \to p^*(E)(-1) \to \mathcal{O}_Y \to 0.$$

This gives a relation,

$$[\mathcal{O}_Y] - [p^*E]\xi^{-1} + \cdots + (-1)^r [p^*(\wedge^r E)]\xi^{-r} = 0,$$

which when multiplied by ξ^r, gives a relation as described above. So, to show that any element in $K_0(Y)$ has a represenation as above, suffices to show that it can be represented as linear combination of ξ^n for all n with coefficients from $K_0(X)$.

As in the previous case, let us first prove that any element in $K_0(\mathbb{P}(E))$ can be written as a linear combination of the ξ^i and then show the uniqueness. Again, the test case is when $X = \operatorname{Spec} k$, k a field and then we are looking at $P = \mathbb{P}^{r-1}$. If H is a hyperplane, we have by localisation, an exact sequence,

$$K_0(H) \to K_0(\mathbb{P}^{r-1}) \to K_0(\mathbb{A}^{r-1}) \to 0.$$

The last term, from the previous step is just $\mathbb{Z}$, generated by $\mathcal{O}$. Then by an easy induction, we see that $K_0(\mathbb{P}^{r-1}$ is generated by $\mathcal{O}_{L_i}$, where L_i is a linear subspace of dimension i. By the Koszul resolution for $\mathcal{O}_{L_i}$, one immediately gets the result.

For the general case, using lemmas [1,2], coupled with the localization sequence [property 1.5] for induction on the dimension of X, one can reduce to the case when $X = \operatorname{Spec} R$, R an integral domain and E is a free module over R. If F is a coherent sheaf over R, then it corresponds to a finitely generated graded module M over S, the polynomial ring in r variables over R. We have a graded resolution of $M \times_R K$, where K is the fraction fileld of R, F_*. Clearly, by clearing denomiantors, we see that there exists an $0 \neq f \in R$, such that M_f has a finite free graded resolution over S_f. This translates into a resolution of F over $U = \operatorname{Spec} R_f \times \mathbb{P}^{r-1}$ by vector bundles which are direct sums of ξ^m for varying m. Thus, we see that in $K_0(U)$, F can be written as claimed. By induction, we know that $K_0(\operatorname{Spec} R/fR) \to K_0(Y - U)$ is onto. Thus by localization, we are done. $\square$

Property 1.8 *If X is smooth, then the natural map $K^0(X) \to K_0(X)$ is an isomorphism.*

Proof: Smoothness ensures that every coherent sheaf has a finite resolution by vector bundles and so it is clear that the map is onto. To check isomorphism, we exhibit an inverse. If F is a coherent sheaf, let E_* be a finite resolution of F. That is, we have,

$$0 \to E_n \to E_{n-1} \to \cdots \to E_1 \to E_0 \to F \to 0$$

exact for some n with E_i's vector bundles over X. Thus, we may define a map $F_{coh} \to K^0(X)$ by defining

$$F \mapsto \sum_{i=0}^{n} [E_i].$$

By elementary homological algebra, one can check that this map is independent of the resolution and it factors through $K_0(X)$. Finally, we note that if E is a vectorbundle, we may take its resolution to be just itself and thus we see that the above map is indeed an inverse to the natural map $K^0(X) \to K_0(X)$. The reader may supply a proof along this line. Let me give a slight variation below. So, for a (finite) resolution E_* of F, define $\theta : F_{coh} \to K^0(X)$ as above. We will check that this is well defined. That is, if E_*, E'_* are two resolutions of F by vector bundles, then $\sum(-1)^i[E_i] = \sum(-1)^i[E'_i]$. We proceed by induction on the homological dimension of F. If it is zero, then F is a vector bundle. Then for any resolution E_*, it is clear that $\theta(E) = [E]$, independent of the resolution. So, assume that we have proved the result for sheaves of homological dimension $< r$ and let F be a sheaf of homological dimension r. Suffices to show that if $0 \to G \to E_0 \to F \to 0$ and $0 \to G' \to E'_0 \to F \to 0$ are exact with E_0, E'_0 vector bundles, so that homological dimension of $G, G' < r$ and thus $\theta(G), \theta(G')$ are well defined in $K^0(X)$, then $\theta(G) + [E'_0] = \theta(G') + [E_0]$. Consider $0 \to H \to E_0 \oplus E'_0 \to F \to 0$, where the right hand map is just addition. Then homological dimension of $H < r$ and thus $\theta(H)$ is well defined. Let

$$0 \to T_m \to T_{m-1} \to \cdots T_1 \to T_0 \to H \to 0,$$

is a resolution of H. We have a commutative diagram,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & G & \to & E_0 & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & H & \to & E_0 \oplus E'_0 & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & E'_0 & = & E'_0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

If we pull back the resolution of H above via the inclusion $G \subset H$ in the above diagram, we get a resolution of G of the following form.

$$0 \to T_m \to T_{m-1} \to \cdots T_1 \to T'_0 \to G \to 0,$$

with an exact sequence, $0 \to T'_0 \to T_0 \to E'_0 \to 0$. Then, we get,

$$\theta(H) \;=\; \sum_{i=0}^{m}(-1)^i[T_i] = [T_0] + \sum_{i=1}^{m}(-1)^i[T_i]$$

$$= \quad [E_0'] + [T_0'] + \sum_{i=1}^{m} (-1)^i [T_i] = [E_0'] + \theta(G).$$

Symmetrically, we get that $\theta(H) = [E_0] + \theta(G')$, which proves that θ is well-defined. θ respects the relation is standard homological algebra, by constructing a compatible resolution given exact sequences of coherent sheaves. But, let me give a proof as before. So, let $0 \to F \to G \to H \to 0$ be an exact seqence. Proof os by induction on r, the maximum of the homolgical dimensions of F, G, H. If it is zero, then all are vector bundles and then the result is clear. Assume proved for less than r and now assume r as above. If $0 \to F' \to E_1 \to F \to 0$ and $0 \to G' \to E_2 \to G \to 0$ are exact with E_i's vector bundles, then we have a resolution, $0 \to G'' \to E_1 \oplus E_2 \to G \to 0$, given by addition as before on the right. Thus, we get a commutative diagram,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F' & \to & E_1 & \to & F & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G'' & \to & E_1 \oplus E_2 & \to & G & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & H' & \to & E_2 & \to & H & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Since F', G'', H' have all homolgical dimension less than r, we get that $\theta(G'') - \theta(F') - \theta(H') = 0$. On the other hand, we have $\theta(F) = [E_1] - \theta(F')$ etc. Thus,

$$\begin{aligned}
\theta(G) - \theta(F) - \theta(H) &= [E_1 \oplus E_2] - \theta(G'') - [E_1] + \theta(F') - [E_2] + \theta(H') \\
&= ([E_1 \oplus E_2] - [E_1] - [E_2]) - (\theta(G'') - \theta(F') - \theta(H')) \\
&= 0.
\end{aligned}$$

This finishes the proof.

□

2 Chow groups

Next we define Chow groups and state similar properties for it. Fix an integer $k \geq 0$ and consider the free abelian group on all irreducible (reduced) subvarieties $Y \subset X$ of dimension k. These are called k-cycles.

Lemma 6 $\mathrm{ord}_P : K^* \to \mathbb{Z}$ *is a group homomorphism.*

This enables us to describe principal divisors on aribtrary irreducible varieties. So, let Z be a $k+1$-dimensional reduced irreducible variety and let $g \neq 0$ be a rational function on Z. For any $Y \subset Z$, closed subvariety of dimension k, the local ring $\mathcal{O}_{Z,Y} = A$ is a one dimensional local domain with maximal ideal, say P. Also, g is a non-zero element in the fraction field. So, define $\mathrm{ord}_Y(g) = \mathrm{ord}_P(g)$. Thus, we can associate with g, the divisor

$$\operatorname{div} g = \sum_{Y \subset Z,\, Y \text{irreducible}, \dim Y = k} \mathrm{ord}_Y(g) Y.$$

This is in fact a finite sum. That is, $\mathrm{ord}_Y(g) = 0$ for all but finitely many Y's. To verify this, we may remove a hyperplane section from X, since the hyperplane section is just a finite union of irreducible divisors. Thus, we may assume that $X = \operatorname{Spec} A$. By the last lemma, we may assume that $g \in A$. By Krull's principal ideal theorem, there are only finitely many height one primes P containing g. It is clear that $\mathrm{ord}_Q(g) \neq 0$ only if $g \in Q$, Q a height one prime ideal of A.

Now we are ready to describe our relation on the group of k-cycles. This relation, called *rational equivalence* is as follows. For any subvariety $Z \subset X$ of dimension $k+1$ and a non-zero rational function f on Z, we consider $\operatorname{div}(f)$, the corresponding divisor. We put these cycles to be zero for all $k+1$ dimensional subvarieties of X and all non-zero rational functions on Z. The quotient group is called $A_k(X)$, the k^{th} Chow group. A similar construction can be done replacing 'dimension k' subvarieties with 'codimension k' subvarieties and then it is called $A^k(X)$. For reasonable varieties X, say of dimension n, it is clear that $A_k(X) = A^{n-k}(X)$. One usually denotes by $A_*(X) = \oplus A_k(X)$ and $A^*(X) = \oplus A^k(X)$. Sometimes, we may drop the asterisk if it is clear which group we are looking at or if it is irrelevant and simply write $A(X)$.

We need a couple of notions before we state the various properties. Let $Y \subset X$ be *any* subscheme and let Y_1 be an irreducible (reduced) component of Y. Then $A = \mathcal{O}_{Y,Y_1}$ is a zero-dimensional local ring and so we have an integer, $\mathrm{ord}\,(Y, Y_1)$, the length of A. If $Y_i, 1 \leq i \leq n$ are the irreducible components of Y, thus we can define a cycle, $\sum \mathrm{ord}\,(Y, Y_i) Y_i \in A(X)$, called the cycle associated with the subscheme Y and written $[Y]$.

Next let $f : X \to Y$ be a dominant morphism of irreducible varieties. Then we get an inclusion $k(Y) \subset k(X)$ of the rational function fields. If

$\dim X = \dim Y$, then this is a finite extension and otherwise it is an infinite extension. Define $\deg(X/Y) = [k(X) : k(Y)]$ if it is finite and zero otherwise.

We will also assume in the following that a flat map $f : X \to Y$ has the following property. There is an integer n so that for any subvariety $Y' \subset Y$, the irreducible componenets of $f^{-1}(Y')$ have all dimension $n + \dim Y'$, unless it is empty. Sometimes we say that f is flat of relative dimension n. The primary examples of such maps are open immersions (relative dimension 0), $p : E \to X$, where E is a vector bundle of rank n (relative dimension n) and $p : \mathbb{P}(E) \to X$, E as above (relative dimension $n-1$).

Now we are ready to state the properties as above for Chow groups.

Property 2.1 (Proper push-forward) *If $f : X \to Y$ is a proper morphism, we get homomorphisms $f_* : A_k(X) \to A_k(Y)$ defined as follows. For an irreducible variety $Z \subset X$, let $Z' = f(Z)$, which is necessarily a closed subvariety of Y, since f is proper. Define $f_*(Z) = \deg(Z/Z')Z'$.*

Proof: We need to check that the map defined above repects rational equivalence. So, let $W \subset X$ be a $k+1$ dimensional subvariety, $T = f(W)$ and let g be a non-zero rational function on W. There are three possibilities for $\dim T$ namely $k+1, k$ or $< k$. We will treat them separately. Let us look at the last case first. If $\dim T < k$, then for any $Z \subset W$ of dimension k, $f(Z) \subset T$ and thus $\deg(Z/f(Z)) = 0$. So, we see that $f_*(\operatorname{div} g) = 0$. Next let us look at the case when $\dim T = k$. So for any $Z \subset W$ of dimension k, we have either $f(Z) = T$ or $\dim f(Z) < k$. So, if $\operatorname{div} g = \sum a_i Z_i$, we see that its image under f_* is just

$$\sum a_i \deg(Z_i/f(Z_i))f(Z_i) = (\sum_{i|f(Z_i)=T} a_i \deg(Z_i/T))T.$$

Thus, to compute this sum, we may go to the generic point η of T and then W_η is a projective curve over η and the above sum is just the degree of the divisor of g on W_η, which is zero from the theory of curves.

Finally, let us look at the case, $\dim T = k+1$. In this case, $k(W)$ is a finite extension of $k(T)$ and we will show that

$$f_*(\operatorname{div} g) = \operatorname{div} \operatorname{Norm}_{k(W)/k(T)} g,$$

which will finish the proof. Let $D \subset T$ be an irreducible subvariety of dimension k. We need to compute $\operatorname{ord}_D(f_*(\operatorname{div} g))$. Since only varieties of dimension k in $f^{-1}(D)$ can contribute to this, we may localise at D and so let

$A = \mathcal{O}_{T,D}$, a local domain of dimension 1 whose fraction field is $K = k(T)$. If W_i's are the irreducible components of dimension k of $f^{-1}(D) \subset W$, we may semilocalise at these closed subsets and get $B = \mathcal{O}_{W,\cup W_i}$ and whose fraction field is $L = k(W)$. Then, $A \subset B$ is a finite map. Let P be the maximal ideal of A corresponding to D, $Q_1, \ldots, Q_n$ be the maximal ideals of B, corresponding to W_is . $PB \subset \cap Q_i$. Notice that $\mathrm{ord}_{W_i}(\mathrm{div}\, g) = \mathrm{ord}_{B_{Q_i}}(g)$. Thus, to prove the result, we only need to show that

$$\sum [B/Q_i : A/P] \mathrm{ord}_{Q_i}(g) = \mathrm{ord}_P(N_{k(W)/k(T)} g). \tag{1}$$

Notice that both sides of the above equation are additive. i.e. for $0 \neq g, h \in B$, the terms are just sums of the corresponding terms for g, h. Next, if $g \in A$, notice that $N_{L/K}(g) = g^n$ where $n = [k(W) : k(T)]$. From lemma [7, 4), 5)] we see that

$$l_A(B/gB) = n l_A(A/gA) = l_A(A/g^n A) = l_A(A/N_{L/K}(g)).$$

Since $B/gB = \prod B_{Q_i}/gB_{Q_i}$ by Chinese Remainder Theorem, we get that $l_A(B/gB) = \sum l_A(B_{Q_i}/gB_{Q_i})$. Now, by lemma [7, 6)] we get that

$$l_A(B_{Q_i}/gB_{Q_i}) = [B/Q_i : A/P] l_B(B_{Q_i}/gB_{Q_i}).$$

Putting these together, we have,

$$l_A(A/N_{L/K}(g)) = \sum [B/Q_i : A/P] l_{B_{Q_i}}(B_{Q_i}/gB_{Q_i}).$$

This proves 1 in this case. □

Property 2.2 (Flat pull-back) *Let $f : X \to Y$ be a flat morphism of relative dimension n. Then we have a map $f^* : A_k(Y) \to A_{k+n}(X)$ defined by $f^*(Y') = [f^{-1}Y']$, where $Y' \subset Y$ is an irreducible variety of Y of dimension k and $f^{-1}Y'$ is the scheme-theoretic inverse image subscheme of X.*

Proof: We need only check that the map defined above respects rational equivalence. This is straight forward. □

Property 2.3 (Loclaization) *For a closed subscheme $Z \subset X$, we have an exact sequence,*

$$A_k(Z) \to A_k(X) \to A_k(X - Z) \to 0,$$

the maps being the obvious ones.

Proof: It is easy to see that we have maps as described above, since $Z \subset X$ is proper, being a closed immersion and $X - Z \subset X$ is flat of relative dimension 0, being an open immersion. Also, it is easy to see that the composite of the maps is zero. If $D \subset X - Z$ is a closed subvariety of dimension k, then $\overline{D}$, the closure of D in X, is a closed subavriety and it is clear that $\overline{D} \mapsto D$, proving surjectivity on the right. Like in the proof in K_0 case, to show exactness in the middle, we procced as follows. If $W \subset X - Z$ is a $k+1$ dimensional subvariety, $g \neq 0$ a rational function on W suufices to show that the closure of $\operatorname{div} g$ in X gives the zero element in $A_k(X)/A_k(Z)$. It is easy to see that if $\overline{W}$ is the closure of W in X, then the class of the closure of $\operatorname{div} g$ in $A_k(X)/A_k(Z)$ is the same as $\operatorname{div} g$ in $\overline{W}$ and thus zero. $\square$

1. Let E be a vector bundle on X and let $p : E \to X$ be the natual map (flat of relative dimension n, the rank of E). Then the natural map $p^* : A_k(X) \to A_{k+n}(E)$ is an isomorphism.

2. (Intersection) Let D be any Cartier divisor on X, so that $\mathcal{O}(D)$ is a line bundle. Then we get a map, $A_k(X) \to A_{k-1}(X)$ given by 'intersecting' with D. (This requires certain amount of finesse, since we will have to intersect varieties completely contained in D and in some sense the crucial part of intersection theory). We write for an $\alpha \in A_k(X)$, the image in $A_{k-1}(X)$ by $D \cdot \alpha$ and sometimes we may drop the $\cdot$. This will depend only on the rational equivalence class of the divisor D and thus we may write this as $\mathcal{O}(D) \cdot \alpha$ or even more suggestively, as $c_1(D) \cdot \alpha$.

3. Let E be a vector bundle of rank n and $p : \mathbb{P}(E) \to X$ and ξ as before. Then any element in $A(\mathbb{P}(E))$ can be uniquely represented as $\sum_{i=0}^{n} \xi^i \cdot p^*(a_i)$, where $\xi \cdot \alpha$ is defined as above.

3 Chern classes

Next we define Segre and Chern classes. Let E be a vector bundle of rank $n+1$ on X, $p : \mathbb{P}(E) \to X$ and ξ as above. Notice that p is a proper and flat map of relative dimension n. We define the i^{th} Segre class $s_i(E)$ as follows. If $\alpha \in A_k(X)$, define $s_i(E) \cdot \alpha = p_*(\xi^{n+i} \cdot p^*(\alpha)) \in A_{k-i}(X)$. So, these are *not* elements of the Chow group, but a collection of homomorphisms from $A_k \to A_{k-i}$ for each i, k. The segre class map has the following properties.

1. $s_i(E) \cdot \alpha = 0$ if $i < 0$.

2. $s_0(E) = \text{Id}$.

3. If E, F are vector bundles on X, then for $\alpha \in A(X)$ one has $s_i(E) \cdot (s_j(F) \cdot \alpha) = s_j(F) \cdot (s_i(E) \cdot \alpha)$.

4. If $f : X \to Y$ is proper, E a vector bundle on Y and $\alpha \in A(X)$,

$$f_*(s_i(f^*E) \cdot \alpha) = s_i(E) \cdot f_*(\alpha).$$

5. If $f : X \to Y$ is flat, E a vector bundle on Y and $\alpha \in A(Y)$, then,

$$s_i(f^*E) \cdot f^*(\alpha) = f^*(s_i(E) \cdot \alpha).$$

6. If E is a line bundle on X, then $s_i(E) \cdot \alpha = -c_1(E) \cdot \alpha$ for any $\alpha \in A(X)$.

Given a vector bundle E, consider the power series,

$$s_t(E) = \sum s_i(E)t^i = 1 + s_1(E)t + s_2(E)t^2 + \cdots.$$

Define the Chern polynomial (which is apriori a power series) to be the inverse of this power series. That is, $c_t(E) = s_t(E)^{-1}$. So,

$$c_t(E) = \sum c_i(E)t^i = 1 + c_1(E)t + c_2(E)t^2 + \cdots$$

Explicitly,

$$c_0(E) = 1, c_1(E) = -s_1(E), c_2(E) = s_1(E)^2 - s_2(E) \,\text{etc.}$$

Next we state some properties of chern classes.

1. If $i > \operatorname{rank} E$, then $c_i(E) = 0$.

2. $c_i(E) \cdot c_j(F) = c_j(F) \cdot c_i(E)$ for any two bundles E, F.

3. (Projection formula) If $f : X \to Y$ is a proper morphism and E a vector bundle on Y, then,

$$f_*(c_i(f^*E) \cdot \alpha) = c_i(E) \cdot f_*(\alpha),$$

   for $\alpha \in A(X)$.

4. (flat-pull back) If $f : X \to Y$ is flat and E a vector bundle on Y, then,
$$c_i(f^*E) \cdot f^*(\alpha) = f^*(c_i(E) \cdot \alpha),$$
for any $\alpha \in A(Y)$.

5. (Whitney sum) If $0 \to E \to F \to G \to 0$ is an exact sequence of vector bundles on X, then $c_t(F) = c_t(E) \cdot c_t(G)$.

6. (normalization) If E is a line bundle on X with $E \cong \mathcal{O}(D)$ for a divisor D, then $c_1(E) \cdot [X] = [D]$.

7. If E is a vector bundle of rank n on X, $p : \mathbb{P}(E) \to X$ and ξ as usual, then we have a relation,
$$\xi^n + c_1(p^*E)\xi^{n-1} + c_2(p^*E)\xi^{n-2} + \cdots + c_n(p^*E) = 0.$$

Notice that Whitney sum ensures that chern class map is well defined in $K^0(X)$. That is, we can speak unambiguosly of the chern class of an element in $K^0(X)$. Consider a vector bundle of rank n and write the polynomial $\sum c_i(E)T^{n-i}$. Since $c_0 = 1$, this is a monic polynomial. The 'formal' roots of this polynomial are called the chern roots of E. So let $a_1, a_2, \ldots, a_n$ be the chern roots. Then we define the chern character of E to be, $\mathrm{ch}\,(E) = \sum \exp(a_i)$. Notice that this formal expression has terms only involving the chern classes, since it is invariant under the symmetric group acting on the roots. Just to get a feel, let me write a few terms.
$$\mathrm{ch}\,(E) = n + c_1 + \frac{1}{2}(c_1^2 - c_2) + \frac{1}{6}(c_1^3 - 3c_1c_2 + c_3) + \cdots$$

Let E and a_i as above. Define the Todd class of E to be,
$$\mathrm{td}\,(E) = \prod \frac{a_i}{1 - \exp(-a_i)},$$
and as before note that the terms are actually expressible in terms of the chern classes of E.

4 Chow rings of smooth varieties

If X is a smooth variety, we can make $A^*(X)$ into a commutative ring. This is the main theme of *intersection theory* and fairly delicate. If $\alpha \in A^k(X)$ and $\beta \in A^l(X)$, we should construct a suitable element $\alpha \cdot \beta \in A^{k+l}(X)$. For an arbitrary (singular) variety, this is impossible in general. The reason why there is a possibility of making sense of this in the smooth case is the following. We may consider the cycle $\alpha \times \beta \in A^{k+l}(X \times X)$. Now we may try to define an intersection with $X \subset X \times X$, sitting as the diagonal. If X is smooth, then the normal bundle of this diagonal is a vector bundle of rank $n = \dim X$ and the earlier techniques of vector bundles can be brought into play (though, requires a lot more justification and this method was discovered by Fulton and McPherson). This enables one to define an intersection product in $A^*(X)$ making it into a commutative graded ring, with $[X] \in A^0(X)$ as the identity element. Thus, if X is a smooth variety, we may and shall consider chern classes as elements of $A^*(X)$ as follows. $c_i(E)$ will be identified with $c_i(E) \cdot [X] \in A(X)$. With this identification, chern character becomes a group homomorphism $\mathrm{ch} : K^0(X) \to A^*(X)$. If $f : X \to Y$ is a proper map of smooth varieties, we may ask whether f_* and ch commute? That is to say, whether for an element $\alpha \in K^0(X) = K_0(X)$, $(\mathrm{ch} \circ f_*)(\alpha) = (f_* \circ \mathrm{ch}\,)(\alpha)$ in $A(Y)$? This is not true and Grothendieck-Riemann-Roch Theorem tells you how to modify this to get a precise answer. We write T_X for the tangent bundle of a smooth variety X.

Finally we can state the Grothendieck Riemann-Roch Theorem.

Theorem: Let $f : X \to Y$ be a proper morphism of smooth varieties. Then for any $\alpha \in K^0(X)$, we have,

$$\mathrm{ch}\,(f_*\alpha) \cdot \mathrm{td}\,(T_Y) = f_*(\mathrm{ch}\,(\alpha) \cdot \mathrm{td}\,(T_X)),$$

in $A(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let us understand this theorem in the simplest cases. So, let us assume that Y is a point. Then $A(Y) = \mathbb{Z}$ generated by the class of the point. Let X be a smooth projective curve and T_X its tangent bundle. If E is a vector bundle, then

$$f_*([E]) = \mathrm{H}^0(E) - \mathrm{H}^1(E)$$

and thus once we identify $A(Y)$ with $\mathbb{Z}$, we see that

$$f_*([E]) = \chi(E) = \mathrm{h}^0(E) - \mathrm{h}^1(E).$$

Taking $\alpha = 1 \in A(Y)$ in the theorem, we see that,

$$\chi(\mathcal{O}_X) = f_*(\mathrm{td}\,(T_X)).$$

One easily checks from our definition of td, that $\mathrm{td}\,(T_X)$ as an operator is just $1 + \frac{1}{2}c_1$ where $c_1 = c_1(T_X)$. Therefore, we get that $\mathrm{td}\,(T_X)$ as an element of

$$A(X) = A_1(X) \oplus A_0(X) = \mathbb{Z}[X] \oplus \mathrm{Pic} X$$

is just $[X] + \frac{1}{2}c_1(T_X) \cdot [X]$. Since $f_* : A(X) \to A(Y)$ is zero on $A_1(X)$ and just computes the degree of the divisor on $A_0(X)$, we see that $\chi(\mathcal{O}_X) = 1 - g = \frac{1}{2}\deg T_X$. This is something we already knew, since $\deg T_X = 2 - 2g$. Now applying to an arbitrary vector bundle E, we will get the usual Riemann-Roch,

$$\chi(E) = \deg E + r(1 - g)$$

where $r = \mathrm{rank}\, E$.

The proof of Riemann-Roch is usually divided into two steps. Any proper map $f : X \to Y$ as above of smooth varieties can be factored into two maps. A closed embedding $i : X \to Y \times \mathbb{P}^n$ for some n and $p : Y \times \mathbb{P}^n \to Y$, the first projection, so that $f = p \circ i$. Then one proves the theorem for i, p separately, where we have much more control on the situation. It is in the latter case of projection that we will be forced to have denominators and thus end up not in $A(Y)$ but $A(Y) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is freqently useful not to do this, especially when we are interested in torsion elements. This works only for closed embeddings.

First we define the total chern class of a vector bundle E on a smooth variety X to be, $c(E) = (\sum c_i(E))[X]$. This gives, by Whitney sum a homomorphism from $K^0(X) \to A^\times(X)$ where the latter is the multiplicative group consisting of elements of the form $1 + a_1 + a_2 + \cdots$, where $a_i \in A^i(X)$.

5 Riemann-Roch without denominators

Theorem: (Riemann-Roch without denominators) Given non-negative integers d, e, there exists a unique power series

$$P(T_1, \ldots, T_d, U_1, \ldots, U_e) \in \mathbb{Z}[\![T_i, U_j]\!]$$

such that for any closed embedding $f : X \to Y$ of smooth varieties with normal bundle N of rank d and any vector bundle E of rank e on X, we have,

$$c(f_*[E]) = 1 + f_*(P(c_1(N), \ldots, c_d(N), c_1(E), \ldots, c_e(E)).$$

Using this formula, one checks that for any vector bundle E as above on X, $c_q(f_*E) = 0$ for $0 < q < d$ and computing the isobaric element of degree 0 of P, one gets that for any vector bundle of rank e on X, $c_d(f_*E) = (-1)^{d-1}(d-1)!e[X] \in A^d(Y)$.

Next consider the filtration in $K^0(X)$, X non-singular (or $K_0(X)$) given as follows. For any integer $p \geq 0$, let $F^pK^0(X) = F^p$ generated by coherent sheaves whose support has codimension greater than or equal to p. Then we get a filtration,

$$0 \subset F^n \subset F^{n-1} \subset \cdots F^1 \subset F^0 = K^0(X),$$

where $n = \dim X$. Using Riemann-Roch without denominators, one checks that for any $\alpha \in F^p$, $c_i(\alpha) = 0$ for $0 < i < p$. Thus we get homomorphisms, $c_p : F^p/F^{p+1} \to A^p(X)$. We always have natural homomorphisms $\phi^p : A^p \to F^p/F^{p+1}$, which is always onto. Riemann-Roch above implies that the composites, $\phi^p \circ c_p$ and $c_p \circ \phi^p$ are both multiplciation by $(-1)^{p-1}(p-1)!$. In particular, we get that c_1, c_2 are isomorphisms. A non-trivial theorem of Murthy will ensure that c_n for $n = \dim X$ is an isomorphism for an affine variety X over an algebraically closed field.

6 Appendix: Modules of finite length

Before we describe the relations we want to impose on these free abelian groups, let me state some elementary results about modules of finite length. If A is a Noetherian ring and M a finitely generated module over A (neither of these assumptions are necessary, though we will assume it for convenience), we say that M is of *finite length* if M has a filtration, $0 = M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = M$ with $M_i/M_{i+1} \cong A/\mathfrak{M}_i$, where $\mathfrak{M}_i \subset A$ are maximal ideals. Such a filtration, if it exists is called a maximal filtration and n is the length of the filtration. For a module of finite length, we will call the least possible length of all maximal filtrations the length of M, and write $l_A(M)$ or when there is no confusion $l(M)$ to denote this number.

Lemma 7 *1. If M is a module of finite length, then any maximal filtration of M has length $= l(M)$.*

2. *Let A be a ring (as always Noetherian) and let $0 \to M \to N \to P \to 0$ be an exact sequence of modules. Then N is of finite length if and only if M, P are of finite length. In this case, $l(N) = l(M) + l(P)$.*

3. *Let A be a one-dimensional domain and $0 \neq a \in A$. Then for any finitely generated module M, $l(M/aM) < \infty$.*

4. *Let A, a as above and M be a finitely generated torsion free module over A. Then $l(M/a^m M) = ml(M/aM)$.*

5. *Let A, a, M be as above and let rank of M be n. Then $l(M/aM) = nl(A/aA)$.*

6. *Let $(A, P) \subset (B, Q)$ be a map of local rings with $P \subset Q$ and let M be a module over B with $l_B(M) < \infty$. Further assume that $[B/Q : A/P] < \infty$. Then M is a finite length module over A and $l_A(M) = [B/Q : A/P]l_B(M)$.*

Proof: The first one is just Jordan-Hölder Theorem.

For the second let us denote by π the map from $N \to P$. If $l(N) < \infty$ with a maximal filtration $\{N_i\}$, one can see easily that $\{P_i = \pi(N_i)\}$ gives a filtration of P and P_i/P_{i+1} is either $A/\mathfrak{M}$ for a maximal ideal or zero. Thus we can extract a finite maximal filtration for P. Similarly, we get a filtration $N_i \cap M$ for M, showing that $l(M) < \infty$. Now assume that both M, P have finite length and let

$$0 = M_n \subset M_{n-1} \subset \cdots \subset M_0 = M,\ 0 = P_k \subset P_{k-1} \subset \cdots \subset P_0 = P$$

be maximal filtrations. Then, one easily sees that,

$$0 = M_n \subset M_{n-1} \subset \cdots \subset M_0 = M = \pi^{-1}(P_k) \subset \pi^{-1}(P_{k-1}) \subset \cdots$$
$$\subset \pi^{-1}(P_0) = N$$

is a maximal filtration of N and this also shows that $l(M) + l(P) = l(N)$.

$\square$

Let A be a one dimensional domain, $0 \neq a \in A$ and let $P \subset A$ be a maximal ideal. Then by the above lemma, $l(A_P/aA_P) < \infty$ and we define $\operatorname{ord}_P(a) = l(A_P/aA_P)$.

Lemma 8 *Let A, P as above and let K be the fraction field of A.*

1. *Let $0 \neq g \in K$. Then for any represenataion $g = a/b$ with $a, b \in A$, $l(A_P/aA_P) - l(A_P/bA_P)$ is constant, which we will call* $\operatorname{ord}_P(g)$.

2. *Let M be a finitely generated torsion free module over A. If $\phi : M \to M$ is an A-module endomorphism which is injective, then $l(M/\phi(M)) < \infty$ and $l(M/\phi(M)) = \mathrm{ord}_P(\det(\phi))$.*

Proof: Clearly, we may assume that $A = A_P$, by localising. Thus we may assume that A is a local domain with maximal ideal P. If $g = a/b = c/d$, with $a, b, c, d \in A$, then $ad = bc$ and so, $l(A/adA) = l(A/bcA)$. But, we have an exact sequence,

$$0 \to A/aA \xrightarrow{d} A/adA \to A/dA \to 0,$$

and so, $l(A/adA) = l(A/aA) + l(A/dA)$. Similarly, we have $l(A/bcA) = l(A/bA) + l(A/cA)$ and so, we get, $l(A/aA) + l(A/dA) = l(A/bA) + l(A/cA)$. So, we get $l(A/aA) - l(A/bA) = l(A/cA) - l(A/dA)$. $\square$

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