Chapter VII QF
Gersten-Witt Complex

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In this chapter, Section 1 is from the Book of Lam. Rest is from ([TWGI, BW]).

1 Scharlau's Transfer

Let $F \hookrightarrow K$ be an extension of fields.

1. For a $F-$quatratic space (V, B, q), we define a $K-$quatratic space (V_K, B_K, q_K) where
   (a) $V_K = K \otimes V$,
   (b) Define $B_K : V_K \times V_K \longrightarrow K$ by
   $$B_K(k \otimes v, k' \otimes v') := kk'B(v, v') \qquad \forall\ k, k' \in K,\ v, v' \in V.$$
   It means,
   $$B_K\left(\sum k_i \otimes v_i, \sum k'_j \otimes v'_j\right) := \sum k_i k'_j B(v_i, v'_j) \qquad \forall\ k_i, k'_j \in K,\ v_i, v'_j \in V.$$
   It needs a checking that B_K is well defined.

(c) It also means $q_K(k \otimes v) = k^2 q(v)$.

2. Suppose $\{v_1, \ldots, v_n\}$ is a basis of V and $B(q)$ be the matrix of V with respect to this basis. Then, $B(q)$ is also the matrix of V_K with respect to $\{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ of V_K.

3. Let $r : F \hookrightarrow K$ the inclusion. As before $M(F)$ denotes the monoid of isometry classes of quadratic forms over F. Then,

   (a) There is a monoid homomorphism $M(F) \longrightarrow M(K)$.

   (b) This induces a homomorphism of the Grothendieck Witt groups: $\hat{r}^* : \widehat{W}(F) \longrightarrow \widehat{W}(K)$.

   (c) Note $\hat{r}^*(\mathbb{H}_F) = \mathbb{H}_K$. So, $\hat{r}^*$ induces a homomprphism of Witt Groups $r^* : W(F) \longrightarrow W(K)$. We have trhe commutative diagram

$$\begin{tikzcd} M(F) \arrow[r] \arrow[d] & \widehat{W}(F) \arrow[r] \arrow[d, "\hat{r}^*"] & W(F) \arrow[d, "r^*"] \\ M(K) \arrow[r] & \widehat{W}(K) \arrow[r] & W(K) \end{tikzcd}$$

4. This association $F \mapsto W(F), r \mapsto r^*$ defines a functor from the category of fields (with $char \neq 2$) to the category of commutative rings.

5. Note $\widehat{W}(\mathbb{R}) \cong \mathbb{Z} \oplus \mathbb{Z}$, $W(\mathbb{R}) \cong \mathbb{Z}$, $\widehat{W}(\mathbb{C}) \cong \mathbb{Z}$, $W(\mathbb{C}) \cong \mathbb{Z}_2$. So, we can compute the maps:

$$\hat{r}^* : \widehat{W}(\mathbb{R}) \longrightarrow \widehat{W}(\mathbb{C}), \qquad r^* : W(\mathbb{R}) \longrightarrow W(\mathbb{C}).$$

   Among what Lam draws our attention:

   (a) $\hat{r}^*(\langle 1, 1 \rangle) = \mathbb{H}_{\mathbb{C}} \in \widehat{W}(\mathbb{C})$.

   (b) More generally, for

$$a \in \dot{F} \ \ and \ \ K = F(\sqrt{a}), \qquad \hat{r}^*(\langle 1, -a \rangle) = \mathbb{H}_K \in \widehat{W}(K)$$

1.1 Scharlau's Transfer

Let $r : F \hookrightarrow K$ be a field extension. Let $s : K \longrightarrow F$ be a nonzero $F-$linear functional, on the vector space K.

1. Note s is surjective. If $s(1) = 1$ then $sr = 1_F$.

2. Suppose (U, B, q) is a quadratic space over K. The the composition sB is a $F-$bilinear form on U

$$\begin{tikzcd} U \times U \arrow[r, "B"] \arrow[dr, "sB"'] & K \arrow[d, "s"] \\ & F \end{tikzcd}$$

So, (U, B, Q) defines a quadratic form (U, sB, sq) over F, where $(sq)(x) = (sB)(x, x) = s(B(x, x))$.

Proposition 1.1. *If (U, B) is a regular $K-$quadratic space, then (U, sB) is a regular $F-$quadratic space.*

Proof. Soppose $sB(x_0, U) = 0$ for some $x_0 \in U$. So, $s(B(x_0, U)) = 0 \quad \forall y \in U$. Since B is regular $B(x_0, y_0) \neq 0$ for some $y_0 \in U$. For any $c \in K$ we have

$$B\left(x_0, \frac{c}{B(x_0, y_0)} y_0\right) = \frac{c}{B(x_0, y_0)} B(x_0, y_0) = c$$

However, $s(c_0) = 1$ for some $c_0 \in K$. So,

$$s\left(B\left(x_0, \frac{c_0}{B(x_0, y_0)} y_0\right)\right) = s(c_0) = 1.$$

This is a contradiction. The proof is complete. ■

Notations and Comments Let $s : K \longrightarrow F$ be as above. We introduce the notations:

1. If U is the $K-$quadratic space, $s_*(U)$ denote the quadratic $F-$space with bilinear for sB.

2. $s_*(U)$ is called the "Transfer" of U.

3. We also have
$$\dim_F s_*(U) = [K:F] \dim_K U.$$

4. Note $s_*(\langle 1 \rangle_K)$ is the quadtaric space with the $F-$bilnear form $(x,y) \mapsto s(xy)$

5. We apply the above to the trace map $tr_{K/F} : K \longrightarrow K$. Consult: Ex 30, Chapter I.

Frobenius Reciprocity

Theorem 1.2. *Let $r : F \hookrightarrow K$ be a finite field extension and $s : K \twoheadrightarrow F$ be a nonzero linear functional. Let V be a quadratic space over F and U be a quadratic space over K. Then, there are $F-$isometries:*

$$s_*(\hat{r}^* V) \otimes_K U) \cong V \otimes_F s_*(U).$$

In particular, taking $U = \langle 1 \rangle_K$ we have

$$s_*(\hat{r}^* V) \cong V \otimes_F s_*(\langle 1 \rangle_K).$$

Proof. Define

$$s_*(K \otimes_F V) \otimes_K U) \longrightarrow V \otimes_F s_*(U) \quad by \quad (k \otimes_F v) \otimes_K u \mapsto v \otimes_F (ku).$$

It is claimed that f is an isometry. First, it is obvious that f is a $F-$linear isomorphism. For $k, k' \in K$, $u, u' \in U$ and $v, v' \in V$ compute inner products: First, on the codomain

$$\langle f((k \otimes v) \otimes_K u), (k' \otimes v)' \otimes_K u') \rangle_F = \langle v \otimes (ku), v' \otimes (k'u') \rangle_F$$

$$= \langle v, v' \rangle_V \cdot \langle ku, k'u' \rangle_{s*U} = \langle v, v' \rangle_V \cdot s(kk' \langle u, u' \rangle_U).$$

Now, compute inner product on the domain $s_*(\hat{r}^* V) \otimes_K U)$:

$$\langle (k \otimes v) \otimes_K u, (k' \otimes v)' \otimes_K u' \rangle_{s_*(XXX)} = s(\langle k \otimes v, k' \otimes v' \rangle_{K \otimes V} \cdot \langle u, u' \rangle_U)$$

$$= s(kk' \langle v, v' \rangle_V \cdot \langle u, u' \rangle_U) = \langle v, v' \rangle_V s(kk' \langle u, u' \rangle_U).$$

The proof is complete. ∎

Corollary 1.3. *We have $s_*(\mathbb{H}_K)$ is a hyperbolic $F-$space.*

Proof. w ehave

$$s_*(\mathbb{H}_K) = s_*(\hat{r}^*(\mathbb{H}_F) = \mathbb{H}_F \otimes s_*(\langle 1 \rangle_K) = [K : F]\mathbb{H}_F \qquad by \quad I.6.1.$$

The proof is complete. ∎

Remark 1.4. We restate theorem 1.2 and other facts:

1. s_* induces a morphism $s_* : \widehat{W}(K) \longrightarrow \widehat{W}(F)$ of $\widehat{W}(F)-$modules.

2. Similarly, s_* induces a morphism $s_* : W(K) \longrightarrow W(F)$ of $W(F)-$modules.

3. s_* is not ring homomorphism. But this is functorial: Suppoe $F \subseteq K \subseteq L$ are two finite extensions of fields. Let $s : K \longrightarrow F$ and $t : L \longrightarrow K$ be two linear functionals. Then, $(ts)_* = s_*t_*$.

4. Choice of s:

   (a) First, by (1.1), $s_*(\langle 1 \rangle_K)$ is regular. So, s induces an isomorphism $K \longrightarrow K^*$ sending $y \mapsto s(*y)$

   (b) Let $t : K \longrightarrow F$ be another functional. By above, $\exists\, k \in K \quad \ni \quad t(z) = s(kz)\ \forall\ z \in K$.

   (c) So, there is a commutative diagram

$$\begin{tikzcd} \widehat{W}(K) \arrow[r, "\langle k \rangle_K"] \arrow[dr, "t_*"'] & \widehat{W}(K) \arrow[d, "s_*"] \\ & \widehat{W}(F) \end{tikzcd}$$

Proof. Suppose (U, B, q) is a regualr quadratic $K-$space. Then, $(tB)(u, v) = t(B(u, v)) = s(kB(u, v))$. The proof is complete. ∎

Since $\langle k \rangle_K$ is a unit, $s_*(\widehat{W}(K)) = t_*(\widehat{W}(K))$. So, the ideal $s_*(\widehat{W}(K))$ is independent of s. We say $s_*(\widehat{W}(K))$ is the transfer ideal.

We can do the same for the Witt groups and define transfer ideal of $W(F)$.

Corollary 1.5. *Let $r : F \hookrightarrow K$ be a finite field extension. Define $W(K/F)$ by the exact sequence*

$$0 \longrightarrow W(K/F) \longrightarrow W(F) \xrightarrow{r^*} W(K)$$

Let $T \subseteq W(F)$ be the transfer ideal. Then, $TW(K/F) = 0$.

Proof. Let $s : K \longrightarrow F$ be any $F-$linear functional. Then $T = s_*(W(K)$. Let U be any $K-$quadratic form and V be any $F-$quadratic form. Then,

$$s_*(V_K \otimes_K U) = V \otimes_F s_*(U).$$

Suppose $V \in W(K/F)$. Then, V_K is hyperbolic. So, by (1.3), LHS is hyperbolic. The proof is complete. ∎

2 Residue class map

Let (A, m) be a local integral domain with $\dim A = 1$. and $K = Q(A)$ be the field of fractions of A. We define a group homonorphism $W(K) \longrightarrow W(k)$.

1. We define a homonorphism $W(K) \longrightarrow W(k)$.

2. Note there is no map from K to k or conversely.

First, we consider Discrete Valuation rings.

1. So, let (A, π, k) be a DVR and $K = Q(A)$.

2. Any element of $\alpha \in K$ can be written as $\alpha = \lambda\pi^r$, where $\lambda \in A$ is a unit and $r \in \mathbb{Z}$.

3. Define $\partial_1, \partial_2 : W(K) \longrightarrow W(k)$ as follows:

$$\partial_1(\langle\lambda\pi^r\rangle) = \begin{cases} \langle\overline{\lambda}\rangle & if\ r\ is\ even \\ 0 & if\ r\ is\ odd. \end{cases} \quad and \quad \partial_2(\langle\lambda\pi^r\rangle) = \begin{cases} 0 & if\ r\ is\ even \\ \langle\overline{\lambda}\rangle & if\ r\ is\ odd \end{cases}$$

4. First, note ∂_i are well defined on the set of isometry classes one dimensional form.

$$\langle\lambda\pi^r\rangle \cong \langle\mu\pi^s\rangle \Longleftrightarrow \lambda\pi^r = (u^2\pi^{2t})\mu\pi^s \Longrightarrow 2 = s + 2t,\ \lambda = u^2\mu.$$

5. Since, any form is diagonalizable, ∂_i is defined on $W(K)$.

Before we proceed to ptove that this is a groups homomorphism, recall the theorem II.4.3 (the Witt version):

Theorem 2.1. Let $\mathcal{F}$ be the free abelian group generated by $\{e_a : a \in \dot{F}\}$. Let $\mathcal{R}$ be the subgroup generated by

1. (R1) $\{e_{ab^2} - e_a : a, b \in \dot{F}\}$

2. (R2) $\{e_a + e_b - e_{a+b} - e_{ab(a+b)} : a, b, a + b \in \dot{F}\}$

3. (R3) $\{e_1 + e_{-1}\}$

Then $W(F) \cong \frac{\mathcal{F}}{\mathcal{R}}$.

Theorem 2.2 (Springer, Knebusch)**.** *We will prove that* ∂_i *are homomorphism of groups.* These are called the residue class homomorphisms

Proof. We will prove that ∂_1 homomorphism of rings. For $a \in \dot{F}$, let e_a be an indeterminate and $\mathcal{G} = \bigoplus \mathbb{Z}[e_a]$ the free abelian group generated by $\{c_a : a \in \dot{F}\}$. define $\partial : \mathcal{G} \longrightarrow W(k)$ exactly like ∂_1. Write

$$a = \lambda_a \pi^{n_a}, b = \lambda_b \pi^{n_b} \quad \textit{and} \quad a + b = \lambda_{a+b} \pi^m.$$

1. (R1) We prove $\partial(e_{ab} - e_a e_b) = 0$.

$$\partial(e_{ab^2} - e_a e_b) = \begin{cases} \text{if } \mathrm{n_a} \text{ is even} & \langle\overline{\lambda_a \lambda_b^2}\rangle - \langle\overline{\lambda_a}\rangle = 0 \\ \text{if } \mathrm{n_a} \text{ is odd} & 0 - 0 = 0 \end{cases}$$

2. (R2) Assume $n_a < n_b$. Then,

$$a + b = (\lambda_a + \lambda_b \pi^{n_b - n_a}) \pi^{n_a} \quad \textit{and} \quad (a+b)ab = (\lambda_a + \lambda_b \pi^{n_b - n_a}) \lambda_a \lambda_b \pi^{2n_a + n_b}$$

Now assume n_a, n_b are even. Then,

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = \langle\overline{\lambda_a}\rangle + \langle\overline{\lambda_b}\rangle - \langle\overline{\lambda_a}\rangle + \langle\overline{\lambda_a^2 \lambda_b}\rangle = 0$$

If both are odd,

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = 0 + 0 + 0 + 0 = 0$$

Now suppose n_a is even and n_b is odd. Then,

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = \langle\overline{\lambda_a}\rangle + 0 - \langle\overline{\lambda_a}\rangle - 0 = 0$$

Suppose n_a is odd and n_b is even. Then,

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = 0 + \langle\overline{\lambda_b}\rangle - 0 - \langle\overline{\lambda_a^2 \lambda_b}\rangle$$

Now consider $n_a = n_b =: n$. So,

$$a + b = (\lambda_a + \lambda_b)\pi^n = \lambda_{a+b}\pi^m \quad \textit{and} \quad (a+b)ab = \lambda_{a+b}\lambda_a\lambda_b\pi^{2n+m}$$

Clearly, $n \leq m$.

1. Suppose $n_a = n_b = n = m$ Then,

$$\lambda_a + \lambda_b = \lambda_{a+b}$$

   (a) Let $n = m$ be odd. Then,

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = 0 + 0 + 0 + 0 = 0$$

   (b) Let $n = m$ be even. Then,

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = \langle\overline{\lambda_a}\rangle + \langle\overline{\lambda_b}\rangle - \langle\overline{\lambda_{a+b}}\rangle - \langle\overline{\lambda_a\lambda_b\lambda_{a+b}}\rangle$$

$$= \langle\overline{\lambda_a}\rangle + \langle\overline{\lambda_b}\rangle - \langle\overline{\lambda_a + \lambda_b}\rangle - \langle\overline{\lambda_a\lambda_b(\lambda_a + \lambda_b)}\rangle = 0 \quad by \quad (R2)$$

2. Now assume $n < m$.In this case $\overline{\lambda_a + \lambda_b} = 0$.

   (a) Let both m, n be odd. Then,

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = 0 + 0 + 0 + 0 = 0$$

   (b) n is even and m is odd. Then,

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = \langle\overline{\lambda_a}\rangle + \langle\overline{\lambda_b}\rangle + 0 + 0 = 0.$$

   (c) n odd and m even.

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = 0 + 0 - \langle\overline{\lambda_{a+b}}\rangle - \langle\overline{\lambda_a\lambda_b\lambda_{a+b})}\rangle$$

$$= 0 + 0 - \langle\overline{\lambda_{a+b}}\rangle + \langle\overline{\lambda_a^2\lambda_{a+b})}\rangle = 0$$

   (d) Both m, n are even. Then,

$$\partial(e_a + e_b - e_{a+b} - e_{(a+b)ab}) = \langle\overline{\lambda_a}\rangle + \langle\overline{\lambda_b}\rangle - \langle\overline{\lambda_{a+b}}\rangle - \langle\overline{\lambda_a\lambda_b\lambda_{a+b}}\rangle$$

$$= (\langle\overline{\lambda_a}\rangle - \langle\overline{\lambda_a}\rangle) - (\langle\overline{\lambda_{a+b}}\rangle - \langle\overline{\lambda_a^2\lambda_{a+b}}\rangle) = 0 - 0 = 0$$

Finally, $\partial(e_1 - e_{-1}) = \langle 1\rangle + \langle -1\rangle = 0$. The theorem follows from theorem 2.1. The proof is complete. ■

2.1 Local domains of diension one

Now suppose (A, m, k) be a local domain with $\dim A = 1$. Write $K = Q(A)$ the field of fractions of A. We define a groups homomorphism $W(K) \longrightarrow W(k)$ as follows.

1. Let B be the integral closure of A, in K. Then,
   (a) $Q(B) = K$, $\dim B = 1$
   (b) B is semilocal. Let $Max(B) = \{m_1, \ldots, m_r\}$.
   (c) B normal. So, B is a Dedekind domain.
   (d) So, B_{m_i} are DVR.
2. Write $k_i = B/m_i$.
3. By theorem 2.2, there are residue class maps (use "∂_1") $\Delta_i : W(K) \longrightarrow W(k_i)$
4. Note $k \hookrightarrow k_i$ is a subfield. Let $s_i : k \longrightarrow k_i$ be any functional. Then, there are transfer homomorphisms $s_{i*} : W(k_i) \longrightarrow W(k)$.
5. Let
$$\Delta = \oplus_{i=1}^m \Delta_i : W(K) \longrightarrow \oplus_{i=1}^m W(k_i)$$
$$\text{and} \quad \Psi = \oplus_{i=1}^m s_{i*} : \oplus_{i=1}^m W(k_i) \longrightarrow W(k).$$
6. Now define a residue class map

$$\partial : W(K) \longrightarrow W(k) \quad \ni \quad \begin{tikzcd} W(K) \arrow[r, "\partial"] \arrow[d, "\Delta"'] & W(k) \\ \oplus_{i=1}^m W(k_i) \arrow[ur, "\Psi"'] & \end{tikzcd} \quad \text{commutes.}$$

2.2 Gersten-Witt Complex

Now suppose A is any intagral domain with $\dim A = d$. Let $X = Spec(A)$. Assume $\forall\ \wp \in Spec(A),\ height(\wp) + \dim\left(\frac{A}{\wp}\right) = d$.

1. Denote $X^{(r)} = \{\wp : height(\wp) = r\}$.
2. $\forall\ \wp \in Spec(A)$, denote $k(\wp) = \left(\frac{A}{\wp}\right)_{\wp}$.
3. First, suppose $\wp \in X^{(1)}$. Then, considering the local ring $A_\wp$ and $K = Q(A)$, there is a residue class homomorphism $W(K) \longrightarrow W(k(\wp))$. Combining all these, there is a homomorphism

$$d_0 : W(K) \longrightarrow \bigoplus_{\wp \in X^{(1)}} W(\wp)$$

   A priory, this is a direct sum of infinitely many homomorphisms. However, given a form $x := \langle x_1, \cdots, x_n\rangle \in W(K)$, the image $d_0(x)$ has only finitely many nonzero components.

4. Similarly, suppose $\wp_r \subseteq \wp_{r+1}$ where $\wp_r \in X^{(r)}$ and $\wp_{r+1} \in X^{(r+1)}$.

   Considering $\dfrac{A_{\wp_{r+1}}}{\wp_r A_{\wp_{r+1}}}$ $\exists$ a residue map $W(k(\wp_r)) \longrightarrow W(k(\wp_{r+1}))$

   Combining all these, there is a homomorphism

$$d_r : \bigoplus_{\wp \in X^{(r)}} W(k(\wp)) \longrightarrow \bigoplus_{\wp \in X^{(r+1)}} W(k(\wp))$$

These homomorphims d_r gives the Gersten-Witt Sequence

$$0 \longrightarrow W(K) \longrightarrow \bigoplus_{\wp\in X^{(1)}} W(k(\wp)) \longrightarrow \bigoplus_{\wp\in X^{(2)}} W(k(\wp)) \longrightarrow \cdots$$

$$\longrightarrow \bigoplus_{\wp\in X^{(r-1)}} W(k(\wp)) \xrightarrow{d_r} \bigoplus_{\wp\in X^{(r)}} W(k(\wp)) \longrightarrow \bigoplus_{\wp\in X^{(r+1)}} W(k(\wp)) \longrightarrow$$

$$\longrightarrow \cdots \longrightarrow \bigoplus_{\wp\in X^{(d)}} W(k(\wp)) \longrightarrow 0$$

Remark 2.3. 1. Same can be defined for any integral scheme X.

2. Unfortunately, we made too many choices. Gersten-Witt Sequence is a complex. However, it is going to be tough to check, because, we went through all the integral closures and all that choices. There is method through Derived Catagory that we discuss in the next section.

3. **Gersten Conjecture for Witt Groups**: In what follows, we assume that (A, m) is a regular local ring. Gersten conjecture states that the sequence

$$0 \longrightarrow W(A) \longrightarrow W(K) \longrightarrow \bigoplus_{\wp \in X^{(1)}} W(k(\wp)) \longrightarrow \cdots$$

$$\longrightarrow \bigoplus_{\wp \in X^{(r)}} W(k(\wp)) \longrightarrow \cdots \longrightarrow \bigoplus_{\wp \in X^{(d)}} W(k(\wp)) \longrightarrow 0$$

is exact. Ojanguren's review in MathSciNet of ([BGPW]) gives a great summary of what is known, as follows.

   (a) Balmer and Pardon independently proved this conjecture, when A is local smooth ring, essentially of finite type, over an infinite field k.

   (b) In ([BGPW]), they proved the same for any regular local ring A, containing a field k (finite or infinite). To do this, they use Popescu's theorem that any regular local ring containing a field, is direct limit of a system of essentially smooth local algebras.

4. Fundamental Ideals: Using the defintion of the fundamantal ideals $I(F) \subseteq W(F)$ of Witt groups $W(F)$, we define the subgroup

$$I_r = \bigoplus_{\wp \in X^{(r)}} I(k(\wp)) \subseteq \bigoplus_{\wp \in X^{(r)}} W(k(\wp))$$

They play a serious role in this theory.

5. There are similar Gersten complexes (conjectures and theorems), using $k-$groups $K_i(k(\wp))$, which I hope to define later.

3 Gersten Complex by Derived categories

Unless stated otherwise, A would be a regular ring with $\dim A = d$ and $1/2 \in A$.

1. Let $\mathcal{P}(A)$ denote the category of finitely generated projective $A-$modules and $D^b(\mathcal{P}(A))$ denote the derived category.
2. Let $\mathcal{F}^r(A) \subseteq D^b(\mathcal{P}(A))$ be the full subcategory of complexes $P_\bullet$ with $Supp(H_i(P_\bullet)) \geq r$. We also write $\mathcal{F}^r := \mathcal{F}^r(A)$, when there no scope of confussion. This gives a filtration
$$D^b(\mathcal{P}(A)) = \mathcal{F}^0(A) \hookleftarrow \mathcal{F}^1(A) \hookleftarrow \cdots \hookleftarrow \mathcal{F}^d(A) \hookleftarrow \mathcal{F}^{d+1}(A) = 0$$
3. Note, all these are triangualted categories with dulity. In fact, $\mathcal{F}^{r+1} \subseteq \mathcal{F}^r$ is a subcategory.

The following is from Balmer's paper ([TWGI]).

3.1 Quotient of Δed categories

Definition 3.1. A sequence

$$0 \longrightarrow J \longrightarrow D \longrightarrow L \longrightarrow 0$$

of (triangulated) categories and fuctors, is said to be exact, if:

1. J is a full subcategory of D.
2. $L = S^{-1}D$ for some multiplicative system S.
3. $\forall$ objects $X \in D \quad X \in J \Longleftrightarrow X \cong 0 \; in \; L$

1. In this case, we say
   (a) J is the kernel category of the functor $D \longrightarrow L$.

(b) We denote $\frac{D}{J} := S^{-1}D$.

2. The choice of the multiplicative system S may not be unique. However, we can enlarge S to S' by defining

$$S' = \{f \in Mor(D) : f \text{ is isomorphism in } L\}.$$

We say that S' is saturated.

3. If D has a duality, with $\#(S) = S$, then $L = S^{-1}D$ inherits the duality. In such cases, we assume that J also inherits the duality.

Following lemma will be of some use later.

Lemma 3.2. *Suppose K is a triangulated category and Z be an object in K. Then $X \mapsto Hom(X, Z)$ and $X \mapsto Hom(Z, X)$ are "exact". That means given any exact triangle*

$$A \xrightarrow[u]{} B \xrightarrow[v]{} C \xrightarrow[w]{} TA$$

the sequences

$$\cdots \longrightarrow Hom(Z, A) \longrightarrow Hom(Z, B) \longrightarrow Hom(Z, C) \longrightarrow Hom(Z, TA) \longrightarrow \cdots$$

and the other one are exact. In the paper of Balmer ([TWGI]), this is referred to as weak kernel and weak cokernel properties.

Proof. Using rotation property, it is enough to prove

$$Hom(Z, A) \longrightarrow Hom(Z, B) \longrightarrow Hom(Z, C) \quad \text{is exact.}$$

Suppose $f \in Hom(Z, B) \mapsto 0$. That means, $vf = 0$. We have the commutative diagram

$$\begin{tikzcd} Z \arrow[r, equal] \arrow[d, dashed, "g"'] & Z \arrow[r] \arrow[d, "f"] \arrow[dr, "0"] & 0 \arrow[r] \arrow[d, "0"] & TZ \arrow[d, dashed, "Tg"] \\ A \arrow[r, "u"'] & B \arrow[r, "v"'] & C \arrow[r, "w"'] & TA \end{tikzcd}$$

Since both rwos are exact, existance of g is given by TR3. Exactness of the other sequence is proved similarly. ∎

Theorem 3.3. *Suppose D is a Δed category and let $J \subseteq D$ be a triangulated full subcategory of D. Then, $\frac{D}{J}$ exists. That means, there is a multiplicative system S such that the definition is satisfied.*

Proof. Define $S = \{f \in Mor(D) : cone(f) \in J\}$. We check, S is a multiplicative system.

1. It follows from the octahedron, that S is closed under composition.
2. (Ore condition) Let

$$\begin{tikzcd} & X \arrow[d, "f"] \\ Z \arrow[r, "s"'] & Y \end{tikzcd} \quad \text{be given} \quad with\ s \in S$$

   Then, s sits on an exact triangle (s, u, δ) and also uf sits on a exact triangle (t, uf, v) as follows

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "g"'] & X \arrow[r, "uf"] \arrow[d, "f"] & C \arrow[r, "v"] \arrow[d, equal] & T[W] \arrow[d] \\ Z \arrow[r, "s"'] & Y \arrow[r, "u"'] & C \arrow[r, "\delta"'] & T[Z] \end{tikzcd}$$

   Since $s \in S$, $C \in J$. Since C is also the cone of t, $t \in J$.

   Likewise, let

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "g"'] & X \\ Z & \end{tikzcd} \quad \text{be given} \quad with\ t \in S$$

   Let C be the cone of t as follows

$$\begin{tikzcd} W \arrow[r, "t"] \arrow[d, "g"'] & X \arrow[r, "u"] & C \arrow[r] & T[W] \\ Z & & & \end{tikzcd} . \quad \text{Now rotate} \quad \begin{tikzcd} T^{-1}C \arrow[r, "v"] & W \arrow[r, "t"] \arrow[d, "g"'] & X \arrow[r, "u"] & C \\ & Z & & \end{tikzcd}$$

Take a cone over gv and complete the diagram

$$\begin{tikzcd}
T^{-1}C \arrow[r, "v"] \arrow[d, equal] & W \arrow[r, "t"] \arrow[d, "g"'] & X \arrow[r, "u"] \arrow[d, dashed, "f"] & C \arrow[d, equal] \\
T^{-1}C \arrow[r, "gv"'] & Z \arrow[r, "s"'] & U \arrow[r] & C
\end{tikzcd}$$

Since cone of s is also $C \in J$, $s \in S$. So, Ore condition is satisfied.

3. (Cancellation) Suppose $sf = sg$ for some $s : Y \longrightarrow Y' \in S$. Write $h = f - g$, then $sh = 0$. Embed s on an exact triangle as follows and consider:

$$\begin{tikzcd}
& X \arrow[dl, dashed, "\gamma"'] \arrow[d, "h"] \arrow[dr, "0"] & & \\
Z \arrow[r, "u"'] & Y \arrow[r, "s"'] & Y' \arrow[r, "\delta"'] & TZ
\end{tikzcd}$$

where $Z \in J$.

Exisitance of γ is given by weak kernel property (see lemma 3.2). Now, embed γ on an exact triangle

$X' \xrightarrow[t]{} X \xrightarrow[\gamma]{} Z \xrightarrow[w]{} TX'$. Since Z ∈ S it follows t ∈ S.

Since $\gamma t = 0$, we have $ht = u\gamma t = 0$. So, one implication of cancellation is established. Similarly, the other implication is established.

So, S is a multiplicative system.

Remains to prove: $X \in J \iff 0 \cong X \in S^{-1}D$. Suppose $X \in J$. Since

$$0 \longrightarrow X = X \longrightarrow 0 \quad \text{is exact} \quad 0 \longrightarrow X \in S$$

So, $0 \cong X \in S^{-1}D$.

Converse: I cannot see it directly. But we can define $\tilde{S} = \{s \in Mor(D) : f^{-1} \in S^{-1}D\}$. (This process is called "saturating"). Then, $S^{-1}D = \tilde{S}^{-1}D$. Therefore, $0 \cong X \in \tilde{S}^{-1}D$ implies $0 \longrightarrow X \in \tilde{S}$. So, the lemma is established with the multiplicative set $\tilde{S}$. ∎

Theorem 3.4 ([BW])**.** Let A be a regular ring with $\dim A = d$. Let Then $\frac{\mathcal{F}^r(A)}{\mathcal{F}^{r+1}(A)}$ exists by theorem 3.3.

1. In fact,
$$\frac{\mathcal{F}^r(A)}{\mathcal{F}^{r+1}(A)} \cong \coprod_{\wp \in X^{(r)}} \mathcal{F}^r(A_\wp) \quad \text{is an equivalence}$$
The equivalence functor is given by pointwise localization.

2. So,
$$0 \longrightarrow \mathcal{F}^{r+1} \longrightarrow \mathcal{F}^r \xrightarrow{\eta} \coprod_{\wp \in X^{(r)}} \mathcal{F}^r(\mathcal{P}(A_\wp)) \longrightarrow 0$$
is an exact sequence of Δed categories, Here η is obtained by pointwise localization.

3. Also, a choice of the multiplicative system is
$$S = \{\varphi \in Hom_{\mathcal{F}^r}(P_\bullet, Q_\bullet) : \forall\ \wp \in X^{(r)}\ \varphi_\wp\ \textit{is a quasi isomorphism}\}$$

Proof. We only need to prove (1). See [BW, Theorem 7.1]. The proof is fairly understandable. But, it uses regularity. $\blacksquare$.

Now, we have the following theorem of Balmer ([TWGI]).

Theorem 3.5 (Balmer)**.** *Suppose*
$$0 \longrightarrow J \xrightarrow{f} D \xrightarrow{g} L \longrightarrow 0$$
be an exact sequence of triangulated categories with duality. Assume $1/2 \in D$. *Assume* D *satisties* $TR4^+$. *Then, there is a connecting homomorphism* $\partial^n : W^n(L) \longrightarrow W^{n+1}(J)$ *such that the sequence an exact sequence*
$$\cdots \longrightarrow W^{n-1}(L) \xrightarrow{\partial^{n-1}} W^n(J) \xrightarrow{W^n(f)} W^n(D) \xrightarrow{W^n(g)} W^n(L) \xrightarrow{\partial^n} W^{n+1}(J) \longrightarrow \cdots$$
is exact.

Proof. Read [TWGII, Theorem 6.2]. The paper is highly readable. ∎

Recall, $T^2 : D \xrightarrow{\sim} T^4 D$ establishes $4-$periodicity of Witt groups. This gives:

Theorem 3.6. *There is a 12 term exact sequence:*

$$\begin{tikzcd}
 & W(J) \arrow[r] & W(D) \arrow[r] & W(L) \arrow[dr] & \\
W^3(L) \arrow[ur] & & & & W^1(J) \arrow[d] \\
W^3(L) \arrow[u] & & & & W^1(D) \arrow[d] \\
W^3(L) \arrow[u] & & & & W^1(L) \arrow[dl] \\
 & W^2(L) \arrow[ul] & W^2(D) \arrow[l] & W^2(J) \arrow[l] &
\end{tikzcd}$$

Proof. follows from theorem 3.5 and the 4-periodicity. ∎

We apply theorem 3.6 to $\mathcal{F}^{r+1}(A) \subseteq \mathcal{F}^r(A)$.

Theorem 3.7. *There is a Gersten-Witt complex constructed as follows, in the proof.*

Proof. Consider, triangulated subcategories

$$\mathcal{F}^{r-1}(A) \subseteq \mathcal{F}^r(A), \quad \mathcal{F}^r(A) \subseteq \mathcal{F}^{r+1}(A), \quad \mathcal{F}^{r+1}(A) \subseteq \mathcal{F}^{r+2}(A).$$

We arrange part of the three Witt exact sequences, given by theorem 3.5 as follows:

$$\begin{tikzcd}
 & W^r\left(\frac{\mathcal{F}^{r-1}(A)}{\mathcal{F}^r(A)}\right) \arrow[d, "\partial"] \arrow[dr, dashed, "q\partial"] & & W^{r+1}(\mathcal{F}^{r+2}(A)) \arrow[d, "\iota"] \\
W^r(\mathcal{F}^{r+1}(A)) \arrow[r, "\iota"] & W^r(\mathcal{F}^r(A)) \arrow[r, "q"] \arrow[d, "\iota"] & W^r\left(\frac{\mathcal{F}^r(A)}{\mathcal{F}^{r+1}(A)}\right) \arrow[r, "\partial"] \arrow[dr, dashed, "q\partial"] & W^{r+1}(\mathcal{F}^{r+1}(A)) \arrow[d, "\partial"] \\
 & W^{r+1}(\mathcal{F}^r(A)) & & W^{r+1}\left(\frac{\mathcal{F}^{r+1}(A)}{\mathcal{F}^{r+2}(A)}\right)
\end{tikzcd}$$

We have the following:

1. By ([BW, Theorem 6.1]),

$$W^r(A_\wp) \cong W(k(\wp)) \qquad \text{(using regularity)}.$$

2. Therefore, by theorem 3.4

$$W^r\left(\frac{\mathcal{F}^r(A)}{\mathcal{F}^{r+1}(A)}\right) \cong \oplus_{\wp \in X^{(r)}} W^r(A_\wp) \cong \oplus_{\wp \in X^{(r)}} W^r(k(\wp)), \text{ and likewise } \cdots$$

So, the diagonal line above gives the complex

$$\cdots \longrightarrow \oplus_{\wp \in X^{(r-1)}} W(k(\wp)) \longrightarrow \oplus_{\wp \in X^{(r)}} W(k(\wp)) \longrightarrow \oplus_{\wp \in X^{(r+1)}} W(k(\wp)) \longrightarrow \cdots$$

■

Remark. It is not clear that the differentials in these two constructions of Gersten-Witt complex agree or not. There is a comment to that effect in ([BW, pp 4-5]).

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