(b) Show that X is not rational by examining $\mathrm{Br}_{tr}(k(X))$.
[*Hint:* Consider the class of the quaternion algebra $(-1, x)$.]

11. ('No-name lemma' for finite groups) Let G be a finite group, and let V and W be vector spaces over a field k, of dimensions n and m, respectively.
(a) Prove that $k(V \oplus W)^G \cong k(V)^G(t_1, \ldots, t_m)$ for some independent variables t_i. [*Hint:* Apply Speiser's lemma to the extension $k(V)|k(V)^G$ and the vector space $W \otimes_k k(V)$.]
(b) Conclude that $k(V)^G(t_1, \ldots, t_m) \cong k(W)^G(u_1, \ldots, u_n)$ for some independent variables t_i and u_j, and hence $\mathrm{Br}_{nr}(k(V)^G) \cong \mathrm{Br}_{nr}(k(W)^G)$.
[*Remark:* This exercise shows that the answer to Problem 6.6.3 depends only on the group G, and not on the representation V.]

12. This exercise gives another proof of the Steinberg relation for Galois cohomology by using Theorem 6.9.1. Let k be a field, m an integer invertible in k, and $k(t)$ the rational function field.
(a) Verify the relation $(t) \cup (1-t) = 0$ in $H^2(k(t), \mu_m^{\otimes 2})$ by calculating the residues of both sides and specializing at 0.
(b) Given $a \in k^\times$, $a \neq 0, 1$, deduce by specialization that $(a) \cup (1-a) = 0$ in $H^2(k, \mu_m^{\otimes 2})$.

7

Milnor K-theory

In this chapter we study the Milnor K-groups introduced in Chapter 4. There are two basic constructions in the theory: that of *tame symbols*, which are analogues of the residue maps in cohomology, and *norm maps* that generalize the field norm $N_{K|k} : K^\times \to k^\times$ for a finite extension $K|k$ to higher K-groups. Of these the first is relatively easy to construct, but showing the well-definedness of the second involves some rather intricate checking. This foreshadows that the chapter will be quite technical, but nevertheless it contains a number of interesting results. Among these, we mention Weil's reciprocity law for the tame symbol over the function field of a curve, a reciprocity law of Rosset and Tate, and considerations of Bloch and Tate about the Bloch–Kato conjecture.

Most of the material in this chapter stems from the three classic papers of Milnor [1], Bass–Tate [1] and Tate [4]. Kato's theorem on the well-definedness of the norm map appears in the second part of his treatise on the class field theory of higher dimensional local fields (Kato [1]), with a sketch of the proof.

7.1 The tame symbol

Recall that we have defined the Milnor K-groups $K_n^M(k)$ attached to a field k as the quotient of the n-th tensor power $(k^\times)^{\otimes n}$ of the multiplicative group of k by the subgroup generated by those elements $a_1 \otimes \cdots \otimes a_n$ for which $a_i + a_j = 1$ for some $1 \leq i < j \leq n$. Thus $K_0^M(k) = \mathbf{Z}$ and $K_1^M(k) = k^\times$. Elements of $K_n^M(k)$ are called *symbols*; we write $\{a_1, \ldots, a_n\}$ for the image of $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(k)$. The relation $a_i + a_j = 1$ will be often referred to as the *Steinberg relation*.

Milnor K-groups are functorial with respect to field extensions: given an inclusion $\phi : k \subset K$, there is a natural map $i_{K|k} : K_n^M(k) \to K_n^M(K)$ induced by ϕ. Given $\alpha \in K_n^M(K)$, we shall often abbreviate $i_{K|k}(\alpha)$ by α_K.

There is also a natural product structure

$$K_n^M(k) \times K_m^M(k) \to K_{n+m}^M(k), \quad (\alpha, \beta) \mapsto \{\alpha, \beta\} \tag{1}$$

coming from the tensor product pairing $(k^\times)^{\otimes n} \times (k^\times)^{\otimes m} \to (k^\times)^{\otimes n+m}$ which obviously preserves the Steinberg relation. This product operation equips the

direct sum

$$K_*^M(k) = \bigoplus_{n\geq 0} K_n^M(k)$$

with the structure of a graded ring indexed by the nonnegative integers. The ring $K_*^M(k)$ is commutative in the graded sense:

Proposition 7.1.1 *The product operation (1) is graded-commutative, i.e. it satisfies*

$$\{\alpha, \beta\} = (-1)^{mn}\{\beta, \alpha\}$$

for $\alpha \in K_n^M(k)$, $\beta \in K_m^M(k)$.

For the proof we first establish an easy lemma:

Lemma 7.1.2 *The group* $K_2^M(k)$ *satisfies the relations*

$$\{x, -x\} = 0 \quad \text{and} \quad \{x, x\} = \{x, -1\}.$$

Proof For the first relation, we compute in $K_2^M(k)$

$$\{x, -x\} + \{x, -(1-x)x^{-1}\} = \{x, 1-x\} = 0,$$

and so

$$\{x, -x\} = -\{x, -(1-x)x^{-1}\} = -\{x, 1-x^{-1}\} = \{x^{-1}, 1-x^{-1}\} = 0.$$

The second one follows by bilinearity. □

Proof of Proposition 7.1.1 By the previous lemma, in $K_2^M(k)$ we have the equalities

$$0 = \{xy, -xy\} = \{x, -x\} + \{x, y\} + \{y, x\} + \{y, -y\} = \{x, y\} + \{y, x\},$$

which takes care of the case $n = m = 1$. The proposition follows from this by a straightforward induction. □

These basic facts are already sufficient for calculating the following example.

Example 7.1.3 For a finite field $\mathbf{F}$ the groups $K_n^M(\mathbf{F})$ are trivial for all $n > 1$.

To see this it is enough to treat the case $n = 2$. Writing ω for a generator of the cyclic group $\mathbf{F}^\times$, we see from bilinearity of symbols that it suffices to show $\{\omega, \omega\} = 0$. By Lemma 7.1.2 this element equals $\{\omega, -1\}$ and hence it has order at most 2. We show that it is also annihilated by an odd integer, which will prove the claim. If $\mathbf{F}$ has order 2^m for some m, we have $0 = \{1, \omega\} = \{\omega^{2^m-1}, \omega\} = (2^m - 1)\{\omega, \omega\}$, and we are done. If $\mathbf{F}$ has odd order, then the same counting argument as in Example 1.3.6 shows that we may find elements $a, b \in \mathbf{F}^\times$ that are *not* squares in $\mathbf{F}$ which satisfy $a + b = 1$. But then $a = \omega^k$, $b = \omega^l$ for some odd integers k, l and hence $0 = \{a, b\} = kl\{\omega, \omega\}$, so we are done again.

As we have seen in Chapter 6, a fundamental tool for studying the Galois cohomology of discrete valuation fields is furnished by the residue maps. We now construct their analogue for Milnor K-theory; the construction will at the same time yield specialization maps for K-groups.

Let K be a field equipped with a discrete valuation $v : K^\times \to \mathbf{Z}$. Denote by A the associated discrete valuation ring and by κ its residue field. Once a local parameter π (i.e. an element with $v(\pi) = 1$) is fixed, each element $x \in K^\times$ can be uniquely written as a product $u\pi^i$ for some unit u of A and integer i. From this it follows by bilinearity and graded-commutativity of symbols that the groups $K_n^M(K)$ are generated by symbols of the form $\{\pi, u_2, \ldots, u_n\}$ and $\{u_1, \ldots, u_n\}$, where the u_i are units in A.

Proposition 7.1.4 *For each $n \geq 1$ there exists a unique homomorphism*

$$\partial^M : K_n^M(K) \to K_{n-1}^M(\kappa)$$

satisfying

$$\partial^M(\{\pi, u_2, \ldots, u_n\}) = \{\overline{u}_2, \ldots, \overline{u}_n\} \tag{2}$$

for all local parameters π and all $(n-1)$-tuples $(u_2, \ldots, u_n)$ of units of A, where $\overline{u}_i$ denotes the image of u_i in κ.

Moreover, once a local parameter π is fixed, there is a unique homomorphism

$$s_\pi^M : K_n^M(K) \to K_n^M(\kappa)$$

with the property

$$s_\pi^M(\{\pi^{i_1}u_1, \ldots, \pi^{i_n}u_n\}) = \{\overline{u}_1, \ldots, \overline{u}_n\} \tag{3}$$

for all n-tuples of integers $(i_1, \ldots, i_n)$ and units $(u_1, \ldots, u_n)$ of A.

The map ∂^M is called the *tame symbol* or the *residue map* for Milnor K-theory; the maps s_π^M are called *specialization maps*. We stress the fact that the s_π^M depend on the choice of π, whereas ∂^M does not, as seen from its definition.

Proof Unicity for s_π^M is obvious, and that of ∂^M follows from the above remark on generators of $K_n^M(K)$, in view of the fact that a symbol of the form $\{u_1, \ldots, u_n\}$ can be written as a difference $\{\pi u_1, u_2, \ldots, u_n\} - \{\pi, u_2, \ldots, u_n\}$ with local parameters π and πu_1, and hence it must be annihilated by ∂^M.

We prove existence simultaneously for ∂^M and the s_π^M via a construction due to Serre. Consider the free graded-commutative $K_*^M(\kappa)$-algebra $K_*^M(\kappa)[x]$ on

one generator x of degree 1. By definition, its elements can be identified with polynomials with coefficients in $K_*^M(\kappa)$, but the multiplication is determined by $\alpha x = -x\alpha$ for $\alpha \in K_1^M(\kappa)$. Now take the quotient $K_*^M(\kappa)[\xi]$ of $K_*^M(\kappa)[x]$ by the ideal $(x^2 - \{-1\}x)$, where $\{-1\}$ is regarded as a symbol in $K_1^M(\kappa)$. The image ξ of x in the quotient satisfies $\xi^2 = \{-1\}\xi$. The ring $K_*^M(\kappa)[\xi]$ has a natural grading in which ξ has degree 1: one has

$$K_*^M(\kappa)[\xi] = \bigoplus_{n \geq 0} L_n,$$

where $L_n = K_n^M(\kappa) \oplus \xi K_{n-1}^M(\kappa)$ for $n > 0$ and $L_0 = K_0^M(\kappa) = \mathbf{Z}$.

Now fix a local parameter π and consider the group homomorphism

$$d_\pi : K^\times \to L_1 = \kappa^\times \oplus \xi\mathbf{Z}$$

given by $\pi^i u \mapsto (\overline{u}, \xi i)$. Taking tensor powers and using the product structure in $K_*^M(\kappa)[\xi]$, we get maps

$$d_\pi^{\otimes n} : (K^\times)^{\otimes n} \to L_n = K_n^M(\kappa) \oplus \xi K_{n-1}^M(\kappa).$$

Denoting by $\pi_1 : L_n \to K_n^M(\kappa)$ and $\pi_2 : L_n \to K_{n-1}^M(\kappa)$ the natural projections, put

$$\partial^M := \pi_2 \circ d_\pi^{\otimes n} \quad \text{and} \quad s_\pi^M := \pi_1 \circ d_\pi^{\otimes n}.$$

One sees immediately that these maps satisfy the properties (2) and (3). Therefore the construction will be complete if we show that $d_\pi^{\otimes n}$ factors through $K_n^M(K)$, for then so do ∂^M and s_π^M.

Concerning our claim about $d_\pi^{\otimes n}$, it is enough to establish the Steinberg relation $d_\pi(x)d_\pi(1-x) = 0$ in L_2. To do so, note first that the multiplication map $L_1 \times L_1 \to L_2$ is given by

$$(x, \xi i)(y, \xi j) = (\{x, y\}, \xi\{(-1)^{ij} x^j y^i\}), \tag{4}$$

where apart from the definition of the L_i we have used the fact that the multiplication map $K_0^M(\kappa) \times K_1^M(\kappa) \to K_1^M(\kappa)$ is given by $(i, x) \mapsto x^i$.

Now take $x = \pi^i u$. If $i > 0$, the element $1 - x$ is a unit, hence $d_\pi(1-x) = 0$ and the Steinberg relation holds trivially. If $i < 0$, then $1 - x = (-u + \pi^{-i})\pi^i$ and $d_\pi(1-x) = (-\overline{u}, \xi i)$. It follows from (4) that

$$d_\pi(x)d_\pi(1-x) = (\overline{u}, \xi i)(-\overline{u}, \xi i) = (\{\overline{u}, -\overline{u}\}, \xi\{(-1)^{i^2}\overline{u}^{-i}(-\overline{u})^i\}),$$

which is 0 in L_2. It remains to treat the case $i = 0$. If $v(1-x) \neq 0$, then replacing x by $1 - x$ we arrive at one of the above cases. If $v(1-x) = 0$, i.e. x and $1 - x$ are both units, then $d_\pi(x)d_\pi(1-x) = (\{\overline{u}, 1 - \overline{u}\}, 0 \cdot \xi) = 0$, and the proof is complete. □

Example 7.1.5 The tame symbol $\partial^M : K_1^M(K) \to K_0^M(\kappa)$ is none but the valuation map $v : K^\times \to \mathbf{Z}$. The tame symbol $\partial^M : K_2^M(K) \to K_1^M(\kappa)$ is given by the formula

$$\partial^M(\{a, b\}) = (-1)^{v(a)v(b)} \overline{a^{v(b)} b^{-v(a)}},$$

where the line denotes the image in κ as usual. One checks this using the definition of ∂^M and the second statement of Lemma 7.1.2.

This is the classical formula for the tame symbol in number theory; it has its origin in the theory of the Hilbert symbol.

Remarks 7.1.6

1. The reader may have rightly suspected that tame symbols and specialization maps are not unrelated. In fact, for $\{a_1, \ldots, a_n\} \in K_n^M(K)$ one has the formula

$$s_\pi^M(\{a_1, \ldots, a_n\}) = \partial^M(\{-\pi, a_1, \ldots, a_n\})$$

   for all local parameters π.

   Indeed, if $a_1 = \pi^i u_1$ for some unit u_1 and integer i, one has

$$\{-\pi, a_1, \ldots, a_n\} = i\{-\pi, \pi, a_2, \ldots, a_n\} + \{-\pi, u_1, a_2, \ldots, a_n\},$$

   where the first term on the right is trivial by the first statement in Lemma 7.1.2. Continuing this process, we may eventually assume that all the a_i are units, in which case the formula follows from the definitions.

2. The behaviour of tame symbols under field extensions can be described as follows. Let $L|K$ be a field extension, and v_L a discrete valuation of L extending v with residue field κ_L and ramification index e. Denoting the associated tame symbol by ∂_L^M, one has for all $\alpha \in K_n^M(K)$

$$\partial_L^M(\alpha_L) = e\,\partial^M(\alpha).$$

   To see this, write a local parameter π for v as $\pi = \pi_L^e u_L$ for some local parameter π_L and unit u_L for v_L. Then for all $(n-1)$-tuples $(u_2, \ldots, u_n)$ of units for v one gets

$$\{\pi, u_2, \ldots, u_n\}_L = e\{\pi_L, u_2, \ldots, u_n\} + \{u_L, u_2, \ldots, u_n\},$$

   where the second term is annihilated by ∂_L^M. The formula follows.

We close this section with the determination of the kernel and the cokernel of the tame symbol.

Proposition 7.1.7 *We have exact sequences*

$$0 \to U_n \to K_n^M(K) \xrightarrow{\partial^M} K_{n-1}^M(\kappa) \to 0$$

and

$$0 \to U_n^1 \to K_n^M(K) \xrightarrow{(s_\pi^M, \partial^M)} K_n^M(\kappa) \oplus K_{n-1}^M(\kappa) \to 0,$$

where U_n is the subgroup of $K_n^M(K)$ generated by those symbols $\{u_1, \dots, u_n\}$ where all the u_i are units in A, and $U_n^1 \subset K_n^M(K)$ is the subgroup generated by symbols $\{x_1, \dots, x_n\}$ with x_1 a unit in A satisfying $\overline{x}_1 = 1$.

The proof uses the following lemma, whose elegant proof is taken from Dennis–Stein [1].

Lemma 7.1.8 *With notations as in the proposition, the subgroup U_n^1 is contained in U_n.*

Proof By writing elements of $K^\times$ as $x = u\pi^i$ with some unit u and prime element π one easily reduces the general case to the case $n = 2$ using bilinearity and the relation $\{\pi, -\pi\} = 0$. Then it suffices to show that symbols of the form $\{1 + a\pi, \pi\}$ with some $a \in A$ are contained in U_2.

Case 1 a is a unit in A. Then

$$\{1 + a\pi, \pi\} = \{1 + a\pi, -a\pi\} + \{1 + a\pi, -a^{-1}\} = \{1 + a\pi, -a^{-1}\}$$

by the Steinberg relation, and the last symbol lies in U_n.

Case 2 a lies in the maximal ideal of A. Then

$$\{1 + a\pi, \pi\} = \left\{1 + \frac{1+a}{1-\pi}\pi, \pi\right\} + \{1 - \pi, \pi\} = \left\{1 + \frac{1+a}{1-\pi}\pi, \pi\right\}.$$

Since here the element $(1+a)(1-\pi)^{-1}$ is a unit in A, we conclude by the first case. □

Proof of Proposition 7.1.7 It follows from the definitions that ∂^M and s_π^M are surjective, and that the two sequences are complexes. By the lemma, for the exactness of the first sequence it is enough to check that each element in $\ker(\partial^M)$ is a sum of elements from U_n and U_n^1. Consider the map

$$\psi : K_{n-1}^M(\kappa) \to K_n^M(K)/U_n^1$$

defined by $\{\overline{u}_1, \dots, \overline{u}_{n-1}\} \mapsto \{\pi, u_1, \dots, u_{n-1}\} \bmod U_n^1$, where the u_i are arbitrary liftings of the $\overline{u}_i$. This is a well-defined map, because replacing some u_i by another lifting u_i' modifies $\{\pi, u_1, \dots, u_{n-1}\}$ by an element in U_n^1. Now for $\alpha \in U_n$ we have $(\psi \circ \partial^M)(\alpha) = 0$, and for an element in $\ker(\partial^M)$ of the form $\beta = \{\pi, u_2, \dots, u_n\}$ with the u_i units we have $0 = (\psi \circ \partial^M)(\beta) = \beta \bmod U_n^1$, i.e. $\beta \in U_n^1$. Since the β of this form generate $\ker(\partial^M)$ together with U_n, we are done.

We now turn to the second sequence. Define a map $K_n^M(\kappa) \to U_n/U_n^1$ by sending $\{\overline{u}_1, \dots, \overline{u}_n\}$ to $\{u_1, \dots, u_n\} \bmod U_n^1$, again with some liftings u_i of the $\overline{u}_i$. We see as above that this map is well defined, and moreover it is an inverse to the map induced by the restriction of s_π^M to U_n (which is of course trivial on U_n^1). □

Remark 7.1.9 It follows from the first sequence above (and was implicitly used in the second part of the proof) that the restriction of s_π^M to $\ker(\partial^M)$ is independent of the choice of π.

Corollary 7.1.10 *Assume moreover that K is complete with respect to v, and let $m > 0$ be an integer invertible in κ. Then the pair (s_π, ∂^M) induces an isomorphism*

$$K_n^M(K)/mK_n^M(K) \xrightarrow{\sim} K_n^M(\kappa)/mK_n^M(\kappa) \oplus K_{n-1}^M(\kappa)/mK_{n-1}^M(\kappa).$$

Proof By virtue of the second exact sequence of the proposition it is enough to see that in this case $mU_n^1 = U_n^1$, which in turn needs only to be checked for $n = 1$ by multilinearity of symbols. But since m is invertible in κ, for each unit $u \in U_1^1$ Hensel's lemma (cf. Appendix, Proposition A.5.5) applied to the polynomial $x^m - u$ shows that $u \in mU_1^1$. □

7.2 Milnor's exact sequence and the Bass–Tate lemma

We now describe the Milnor K-theory of the rational function field $k(t)$, and establish an analogue of Faddeev's exact sequence due to Milnor.

Recall that the discrete valuations of $k(t)$ trivial on k correspond to the local rings of closed points P on the projective line $\mathbf{P}_k^1$. As before, we denote by $\kappa(P)$ their residue fields and by v_P the associated valuations. At each closed point $P \neq \infty$ a local parameter is furnished by a monic irreducible polynomial $\pi_P \in k[t]$; at $P = \infty$ one may take $\pi_P = t^{-1}$. The degree of the field extension $[\kappa(P) : k]$ is called the degree of the closed point P; it equals the degree of the polynomial π_P.

By the theory of the previous section we obtain tame symbols

$$\partial_P^M : K_n^M(k(t)) \to K_{n-1}^M(\kappa(P))$$

and specialization maps

$$s_{\pi_P}^M : K_n^M(k(t)) \to K_n^M(\kappa(P)).$$

Note that since each element in $k(t)^\times$ is a unit for all but finitely many valuations v_P, the image of the product map

$$\partial^M := (\partial_P^M) : K_n^M(k(t)) \to \prod_{P \in \mathbf{P}^1_0 \setminus \{\infty\}} K_{n-1}^M(\kappa(P))$$

lies in the direct sum.

Theorem 7.2.1 (Milnor) *The sequence*

$$0 \to K_n^M(k) \to K_n^M(k(t)) \xrightarrow{\partial^M} \bigoplus_{P \in \mathbf{P}^1_0 \setminus \{\infty\}} K_{n-1}^M(\kappa(P)) \to 0$$

is exact and split by the specialization map $s_{t^{-1}}^M$ at ∞.

Note that for $i = 1$ we get the sequence

$$1 \to k^\times \to k(t)^\times \xrightarrow{\partial^M} \bigoplus_{\pi} \mathbf{Z} \to 0$$

which is equivalent to the decomposition of a rational function into a product of irreducible factors.

The proof exploits the filtration on $K_n^M(k(t))$

$$K_n^M(k) = L_0 \subset L_1 \subset \cdots \subset L_d \subset \ldots \tag{5}$$

where L_d is the subgroup of $K_n^M(k(t))$ generated by those symbols $\{f_1, \ldots, f_n\}$ where the f_i are polynomials in $k[t]$ of degree $\leq d$.

The key statement is the following.

Lemma 7.2.2 *For each $d > 0$ consider the homomorphism*

$$\partial_d^M : K_n^M(k(t)) \to \bigoplus_{\deg(P)=d} K_{n-1}^M(\kappa(P))$$

defined as the direct sum of the maps ∂_P^M for all closed points P of degree d. Its restriction to L_d induces an isomorphism

$$\overline{\partial}_d^M : L_d/L_{d-1} \xrightarrow{\sim} \bigoplus_{\deg(P)=d} K_{n-1}^M(\kappa(P)).$$

Proof If P is a closed point of degree d, the maps ∂_P^M are trivial on the elements of L_{d-1}, hence the map $\overline{\partial}_d^M$ exists. To complete the proof we construct an inverse for $\overline{\partial}_d^M$.

Let P be a closed point of degree d. For each element $\overline{a} \in \kappa(P)$ there exists a unique polynomial $a \in k[t]$ of degree $\leq d-1$ whose image in $\kappa(P)$ is $\overline{a}$. Define maps

$$h_P : K_{n-1}^M(\kappa(P)) \to L_d/L_{d-1}$$

by the assignment

$$h_P(\{\overline{a}_2, \ldots, \overline{a}_n\}) = \{\pi_P, a_2, \ldots, a_n\} \mod L_{d-1}.$$

The maps h_P obviously satisfy the Steinberg relation, for $\overline{a}_i + \overline{a}_j = 1$ implies $a_i + a_j = 1$. So if we show that they are linear in each variable, we get that each h_P is a homomorphism, and then by construction the direct sum $\bigoplus h_P$ yields an inverse for $\overline{\partial}_d^M$.

We check linearity in the case $n = 2$, the general case being similar. For $\overline{a}_2 = \overline{b}_2\overline{c}_2$, we compare the polynomials a_2 and b_2c_2. If they are equal, the claim is obvious. If not, we perform Euclidean division of b_2c_2 by π_P to get $b_2c_2 = a_2 - \pi_P f$ with some polynomial $f \in k[t]$ of degree $\leq d-1$ (note that the rest of the division must be a_2 by uniqueness). Therefore

$$\frac{\pi_P f}{a_2} = 1 - \frac{b_2c_2}{a_2}, \tag{6}$$

and so in $K_2^M(k(t))$ we have the equalities

$$\{\pi_P, b_2c_2\} - \{\pi_P, a_2\} = \left\{\pi_P, \frac{b_2c_2}{a_2}\right\} = -\left\{\frac{f}{a_2}, \frac{b_2c_2}{a_2}\right\} + \left\{\frac{\pi_P f}{a_2}, \frac{b_2c_2}{a_2}\right\} = -\left\{\frac{f}{a_2}, \frac{b_2c_2}{a_2}\right\},$$

where we used the equality (6) in the last step. The last symbol lies in L_{d-1}, and the claim follows. □

Proof of Theorem 7.2.1 Using induction on d, we derive from the previous lemma exact sequences

$$0 \to L_0 \to L_d \to \bigoplus_{\deg(P) \leq d} K_{n-1}^M(\kappa(P)) \to 0$$

for each $d > 0$. These exact sequences form a natural direct system with respect to the inclusions coming from the filtration (5). As $L_0 = K_n^M(k)$ and $\bigcup L_d = K_n^M(k(t))$, we obtain the exact sequence of the theorem by passing to the limit. The statement about $s_{t^{-1}}^M$ is straightforward. □

Note that Milnor's exact sequence bears a close resemblance to Faddeev's exact sequence in the form of Corollary 6.9.3. As in that chapter, the fact that the sequence splits allows us to define *coresidue* maps

$$\psi_P^M : K_{n-1}^M(\kappa(P)) \to K_n^M(k(t))$$

for all closed points $P \neq \infty$, enjoying the properties $\partial_P^M \circ \psi_P^M = \mathrm{id}_{\kappa(P)}$ and $\partial_P^M \circ \psi_Q^M = 0$ for $P \neq Q$. We thus obtain the following formula useful in calculations.

Corollary 7.2.3 *We have the equality*

$$\alpha = s_{t^{-1}}(\alpha)_{k(t)} + \sum_{P \in \mathbf{A}^1_0} (\psi_P^M \circ \partial_P^M)(\alpha)$$

for all $n > 0$ and all $\alpha \in K_n^M(k(t))$.

Note that since each element in $k(t)^\times$ is a unit for all but finitely many valuations v_P, the image of the product map

$$\partial^M := (\partial_P^M) : K_n^M(k(t)) \to \prod_{P \in \mathbf{P}_k^1 \setminus \{\infty\}} K_{n-1}^M(\kappa(P))$$

lies in the direct sum.

Theorem 7.2.1 (Milnor) *The sequence*

$$0 \to K_n^M(k) \to K_n^M(k(t)) \xrightarrow{\partial^M} \bigoplus_{P \in \mathbf{P}_0^1 \setminus \{\infty\}} K_{n-1}^M(\kappa(P)) \to 0$$

is exact and split by the specialization map $s_{t^{-1}}^M$ at ∞.

Note that for $i = 1$ we get the sequence

$$1 \to k^\times \to k(t)^\times \xrightarrow{\partial^M} \bigoplus_P \mathbf{Z} \to 0$$

which is equivalent to the decomposition of a rational function into a product of irreducible factors.

The proof exploits the filtration on $K_n^M(k(t))$

$$K_n^M(k) = L_0 \subset L_1 \subset \cdots \subset L_d \subset \dots \tag{5}$$

where L_d is the subgroup of $K_n^M(k(t))$ generated by those symbols $\{f_1, \dots, f_n\}$ where the f_i are polynomials in $k[t]$ of degree $\leq d$.

The key statement is the following.

Lemma 7.2.2 *For each $d > 0$ consider the homomorphism*

$$\partial_d^M : K_n^M(k(t)) \to \bigoplus_{\deg(P)=d} K_{n-1}^M(\kappa(P))$$

defined as the direct sum of the maps ∂_P^M for all closed points P of degree d. Its restriction to L_d induces an isomorphism

$$\overline{\partial}_d^M : L_d / L_{d-1} \xrightarrow{\sim} \bigoplus_{\deg(P)=d} K_{n-1}^M(\kappa(P)).$$

Proof If P is a closed point of degree d, the maps ∂_P^M are trivial on the elements of L_{d-1}, hence the map $\overline{\partial}_d^M$ exists. To complete the proof we construct an inverse for $\overline{\partial}_d^M$.

Let P be a closed point of degree d. For each element $\overline{a} \in \kappa(P)$ there exists a unique polynomial $a \in k[t]$ of degree $\leq d-1$ whose image in $\kappa(P)$ is $\overline{a}$. Define maps

$$h_P : K_{n-1}^M(\kappa(P)) \to L_d / L_{d-1}$$

by the assignment

$$h_P(\{\overline{a}_2, \dots, \overline{a}_n\}) = \{\pi_P, a_2, \dots, a_n\} \mod L_{d-1}.$$

The maps h_P obviously satisfy the Steinberg relation, for $\overline{a}_i + \overline{a}_j = 1$ implies $a_i + a_j = 1$. So if we show that they are linear in each variable, we get that each h_P is a homomorphism, and then by construction the direct sum $\oplus h_P$ yields an inverse for $\overline{\partial}_d^M$.

We check linearity in the case $n = 2$, the general case being similar. For $\overline{a}_2 = \overline{b}_2\overline{c}_2$, we compare the polynomials a_2 and b_2c_2. If they are equal, the claim is obvious. If not, we perform Euclidean division of b_2c_2 by π_P to get $b_2c_2 = a_2 - \pi_P f$ with some polynomial $f \in k[t]$ of degree $\leq d-1$ (note that the rest of the division must be a_2 by uniqueness). Therefore

$$\frac{\pi_P f}{a_2} = 1 - \frac{b_2c_2}{a_2}, \tag{6}$$

and so in $K_2^M(k(t))$ we have the equalities

$$\{\pi_P, b_2c_2\} - \{\pi_P, a_2\} = \left\{\pi_P, \frac{b_2c_2}{a_2}\right\} = -\left\{\frac{f}{a_2}, \frac{b_2c_2}{a_2}\right\} + \left\{\frac{\pi_P f}{a_2}, \frac{b_2c_2}{a_2}\right\} = -\left\{\frac{f}{a_2}, \frac{b_2c_2}{a_2}\right\},$$

where we used the equality (6) in the last step. The last symbol lies in L_{d-1}, and the claim follows. □

Proof of Theorem 7.2.1 Using induction on d, we derive from the previous lemma exact sequences

$$0 \to L_0 \to L_d \to \bigoplus_{\deg(P) \leq d} K_{n-1}^M(\kappa(P)) \to 0$$

for each $d > 0$. These exact sequences form a natural direct system with respect to the inclusions coming from the filtration (5). As $L_0 = K_n^M(k)$ and $\bigcup L_d = K_n^M(k(t))$, we obtain the exact sequence of the theorem by passing to the limit. The statement about $s_{t^{-1}}^M$ is straightforward. □

Note that Milnor's exact sequence bears a close resemblance to Faddeev's exact sequence in the form of Corollary 6.9.3. As in that chapter, the fact that the sequence splits allows us to define *coresidue* maps

$$\psi_P^M : K_{n-1}^M(\kappa(P)) \to K_n^M(k(t))$$

for all closed points $P \neq \infty$, enjoying the properties $\partial_P^M \circ \psi_P^M = \mathrm{id}_{\kappa(P)}$ and $\partial_P^M \circ \psi_Q^M = 0$ for $P \neq Q$. We thus obtain the following formula useful in calculations.

Corollary 7.2.3 *We have the equality*

$$\alpha = s_{t^{-1}}(\alpha)_{k(t)} + \sum_{P \in \mathbf{A}_0^1} (\psi_P^M \circ \partial_P^M)(\alpha)$$

for all $n > 0$ and all $\alpha \in K_n^M(k(t))$.

For all $P \neq \infty$ we define *norm maps* $N_P : K_n^M(\kappa(P)) \to K_n^M(k)$ by the formula

$$N_P := -\partial_\infty^M \circ \psi_P^M$$

for all $n \geq 0$. For $P = \infty$ we define N_P to be the identity map of $K_n^M(k)$.

With the above notations, Milnor's exact sequence implies

Corollary 7.2.4 (Weil reciprocity law) *For all $\alpha \in K_n^M(k(t))$ we have*

$$\sum_{P \in \mathbf{P}_k^1} (N_P \circ \partial_P^M)(\alpha) = 0.$$

Proof For $P \neq \infty$ we have from the defining property of the maps ψ_P

$$\partial_P^M\Big(\alpha - \sum_{P \neq \infty} (\psi_P^M \circ \partial_P^M)(\alpha)\Big) = \partial_P^M(\alpha) - \partial_P^M(\alpha) = 0,$$

so by Milnor's exact sequence

$$\alpha - \sum_{P \neq \infty} (\psi_P^M \circ \partial_P^M)(\alpha) = \beta$$

for some β coming from $K_n^M(k)$. We have $\partial_\infty^M(\beta) = 0$, so the corollary follows by applying ∂_∞^M to both sides. □

Weil's original reciprocity law concerned the case $n = 2$ and had the form

$$\sum_{P \in \mathbf{P}_0^1} (N_{\kappa(P)|k} \circ \partial_P^M)(\alpha) = 0.$$

Note that in this case the tame symbols ∂_P^M have an explicit description by Example 7.1.5. To relate this form to the previous corollary, it suffices to use the second statement of the following proposition.

Proposition 7.2.5 *For $n = 0$ the map $N_P : K_0^M(\kappa(P)) \to K_0^M(k)$ is given by multiplication with $[\kappa(P) : k]$, and for $n = 1$ it coincides with the field norm $N_{\kappa(P)|k} : \kappa(P)^\times \to k^\times$.*

The proof relies on the following behaviour of the norm map under extensions of the base field.

Lemma 7.2.6 *Let $K|k$ be a field extension, and P a closed point of $\mathbf{P}_k^1$. Then the diagram*

$$\begin{array}{ccc} K_n^M(\kappa(P)) & \xrightarrow{N_P} & K_n^M(k) \\ \big\downarrow{\scriptstyle \oplus i_{\kappa(Q)|\kappa(P)}} & & \big\downarrow{\scriptstyle i_{K|k}} \\ \bigoplus_{Q \mapsto P} K_n^M(\kappa(Q)) & \xrightarrow{\sum e_Q N_Q} & K_n^M(K) \end{array}$$

commutes, where the notation $Q \mapsto P$ stands for the closed points of $\mathbf{P}_K^1$ lying above P, and e_Q is the ramification index of the valuation v_Q extending v_P to $K(t)$.

Proof According to Remark 7.1.6 (2), the diagram

$$\begin{array}{ccc} K_{n+1}^M(k(t)) & \xrightarrow{\partial_P^M} & K_n^M(\kappa(P)) \\ \big\downarrow{\scriptstyle i_{K(t)|k(t)}} & & \big\downarrow{\scriptstyle \oplus e_Q i_{\kappa(Q)|\kappa(P)}} \\ K_{n+1}^M(K(t)) & \xrightarrow{\oplus \partial_Q^M} & \bigoplus_{Q \mapsto P} K_n^M(\kappa(Q)) \end{array}$$

commutes. Hence so does the diagram

$$\begin{array}{ccc} K_{n+1}^M(k(t)) & \xleftarrow{\psi_P^M} & K_n^M(\kappa(P)) \\ \big\downarrow{\scriptstyle i_{K(t)|k(t)}} & & \big\downarrow{\scriptstyle \oplus i_{\kappa(Q)|\kappa(P)}} \\ K_{n+1}^M(K(t)) & \xleftarrow{\sum e_Q \psi_Q^M} & \bigoplus_{Q \mapsto P} K_n^M(\kappa(Q)) \end{array}$$

whence the compatibility of the lemma in view of the definition of the norm maps N_P. □

Proof of Proposition 7.2.5 Apply the above lemma with K an algebraic closure of k. In this case the points Q have degree 1 over K, so the maps N_Q are identity maps. Moreover, the vertical maps are injective for $n = 0, 1$. The statement for $n = 0$ then follows from the formula $\sum e_Q = [\kappa(P) : k]$ (a particular case of Proposition A.6.7 of the Appendix), and for $n = 1$ from the definition of the field norm $N_{\kappa(P)|k}(\alpha)$ as the product of the roots in K (considered with multiplicity) of the minimal polynomial of α. □

Remark 7.2.7 For later use, let us note that the norm maps N_P satisfy the *projection formula:* for $\alpha \in K_n^M(k)$ and $\beta \in K_m^M(\kappa(P))$ one has

$$N_P(\{\alpha_{\kappa(P)}, \beta\}) = \{\alpha, N_P(\beta)\}.$$

This is an immediate consequence of the definitions.

We conclude this section by a very useful technical statement which is not a consequence of Milnor's exact sequence itself, but is proven in a similar vein. Observe that if $K|k$ is a field extension, the graded ring $K_*^M(K)$ becomes a (left) $K_*^M(k)$-module via the change-of-fields map $K_*^M(k) \to K_*^M(K)$ and the product structure.

Proposition 7.2.8 (Bass–Tate Lemma) *Let $K = k(a)$ be a field extension obtained by adjoining a single element a of degree d to k. Then $K_*^M(K)$ is generated as a left $K_*^M(k)$-module by elements of the form*

$$\{\pi_1(a), \pi_2(a), \dots, \pi_m(a)\},$$

where the π_i are monic irreducible polynomials in $k[t]$ satisfying $\deg(\pi_1) < \deg(\pi_2) < \cdots < \deg(\pi_m) \le d-1$.

The proof is based on the following property of the subgroups L_d introduced in Lemma 7.2.2.

Lemma 7.2.9 *The subgroup $L_d \subset K_n^M(k(t))$ is generated by symbols of the shape*

$$\{a_1, \dots, a_m, \pi_{m+1}, \pi_{m+2}, \dots, \pi_n\}, \tag{7}$$

where the a_i belong to $k^\times$ and the π_i are monic irreducible polynomials in $k[t]$ satisfying $\deg(\pi_{m+1}) < \deg(\pi_{m+2}) < \cdots < \deg(\pi_n) \le d$.

Proof By factoring polynomials into irreducible terms and using bilinearity and graded-commutativity of symbols, we obtain generators for the group L_d of the shape (7), except that the π_i a priori only satisfy $\deg(\pi_{m+1}) \le \cdots \le \deg(\pi_n) \le d$. The point is to show that the inequalities may be chosen to be strict, which we do in the case $n = 2$ for polynomials π_1, π_2 of the same degree, the general case being similar. We use induction on d starting from the case $d = 0$ where we get constants $\pi_1 = a_1$, $\pi_2 = a_2$. So assume $d > 0$. If $\deg(\pi_1) = \deg(\pi_2) < d$, we are done by induction. It remains the case $\deg(\pi_1) = \deg(\pi_2) = d$, where we perform Euclidean division to get $\pi_2 = \pi_1 + f$ with some f of degree $\le d-1$. So $1 = \pi_1/\pi_2 + f/\pi_2$ and therefore $\{\pi_1/\pi_2, f/\pi_2\} = 0$ in $K_2^M(k(t))$. Using Lemma 7.1.2 we may write

$$\{\pi_1, \pi_2\} = \{\pi_1/\pi_2, \pi_2\} + \{\pi_2, -1\} = -\{\pi_1/\pi_2, f/\pi_2\} + \{\pi_1/\pi_2, f\} + \{\pi_2, -1\},$$

which equals $-(\{f, \pi_1\} + \{-f, \pi_2\})$ by bilinearity and graded-commutativity. We conclude by decomposing the polynomial f into irreducible factors. □

Proof of Proposition 7.2.8 Let π_P be the minimal polynomial of a over k: it defines a closed point P of degree d on $\mathbf{P}^1_k$. It follows from Lemma 7.2.2 that the tame symbol ∂_P^M induces a surjection of L_d onto $K_n^M(\kappa(P))$. Applying the previous lemma, we conclude that $K_n^M(\kappa(P))$ is generated by symbols of the form

$$\partial_P^M\{a_1, \dots, a_m, \pi_{m+1}, \pi_{m+2}, \dots, \pi_n\},$$

where the a_i belong to $k^\times$ and the π_i are monic irreducible polynomials satisfying $\deg(\pi_1) < \deg(\pi_2) < \cdots < \deg(\pi_n) \le d$. If $\pi_n \ne \pi_P$, all the π_i satisfy $v_\pi(\pi_i) = 0$ and the above symbols are zero. For $\pi_n = \pi_P$, they equal $\{a_1, \dots, a_m, \pi_{m+1}(a), \pi_{m+2}(a), \dots, \pi_{n-1}(a)\}$ up to a sign by the defining property of ∂_P^M, and the proposition follows. □

In what follows we shall use the Bass–Tate lemma several times via the following corollary.

Corollary 7.2.10 *Let $K|k$ be a finite field extension. Assume one of the following holds:*

- *$K|k$ is a quadratic extension.*
- *$K|k$ is of prime degree p and k has no nontrivial finite extensions of degree prime to p.*

Then $K_*^M(K)$ is generated as a left $K_*^M(k)$-module by $K_1^M(K) = K^\times$. In other words, the product maps $K_{n-1}^M(k) \otimes K^\times \to K_n^M(K)$ are surjective.

Proof In both cases, K is obtained by adjoining a single element a to k, and the only monic irreducible polynomials in $k[t]$ of degree strictly smaller than $[K:k]$ are the linear polynomials $x - a$. We conclude by applying the proposition. □

Remark 7.2.11 A typical case when the second condition of the corollary is satisfied is when k is a *maximal prime to p extension* of some field $k_0 \subset k$. This is an algebraic extension $k|k_0$ such that all finite subextensions have degree prime to p and which is maximal with respect to this property. If k_0 is perfect or has characteristic p, we can construct such an extension k by taking the subfield of a separable closure k_s of k_0 fixed by a pro-p Sylow subgroup of $\mathrm{Gal}(k_s|k_0)$. If k_0 is none of the above, we may take k to be a maximal prime to p extension of a perfect closure of k_0.

7.3 The norm map

Let $K|k$ be a finite field extension. In this section we construct norm maps $N_{K|k} : K_n^M(K) \to K_n^M(k)$ for all $n \ge 0$ satisfying the following properties:

1. The map $N_{K|k} : K_0^M(K) \to K_0^M(k)$ is multiplication by $[K:k]$.
2. The map $N_{K|k} : K_1^M(K) \to K_1^M(k)$ is the field norm $N_{K|k} : K^\times \to k^\times$.
3. (Projection formula) Given $\alpha \in K_n^M(k)$ and $\beta \in K_m^M(K)$, one has
$$N_{K|k}(\{\alpha_K, \beta\}) = \{\alpha, N_{K|k}(\beta)\}.$$
4. (Composition) Given a tower of field extensions $K'|K|k$, one has
$$N_{K'|k} = N_{K|k} \circ N_{K'|K}.$$

Furthermore, a reasonable norm map should be compatible (for finite separable extensions) with the corestriction maps on cohomology via the Galois symbol. This issue will be discussed in the next section.

Remark 7.3.1 For any norm map satisfying the above properties (1)–(3) the composite maps $N_{K|k} \circ i_{K|k} : K_n^M(k) \to K_n^M(k)$ are given by multiplication with the degree $[K : k]$ for all n. This is obvious for $n = 0, 1$, and the case $n > 1$ follows from the case $n = 1$ by an easy induction using the projection formula.

In the case when $K = k(a)$ is a simple field extension, the minimal polynomial of a defines a closed point P on $\mathbf{P}^1_k$ for which $K \cong \kappa(P)$. The norm map N_P of the previous section satisfies properties (1) and (2) by virtue of Proposition 7.2.5, as well as property (3) by Remark 7.2.7, so it is a natural candidate for $N_{K|k}$. But even in this case one has to check that the definition depends only on K and not on the choice of P.

Changing the notation slightly, for a simple finite field extension $K = k(a)$ define $N_{a|k} : K_n^M(k(a)) \to K_n^M(k)$ by $N_{a|k} := N_P$, where P is the closed point of $\mathbf{P}^1_k$ considered above. Given an arbitrary finite field extension $K|k$, write $K = k(a_1, \dots, a_r)$ for some generators $a_1, \dots, a_r$ and consider the chain of subfields

$$k \subset k(a_1) \subset k(a_1, a_2) \subset \cdots \subset k(a_1, \dots, a_r) = K.$$

Now put

$$N_{a_1,\dots,a_r|k} := N_{a_r|k(a_1,\dots,a_{r-1})} \circ \cdots \circ N_{a_2|k(a_1)} \circ N_{a_1|k}.$$

Note that by the preceding discussion the maps $N_{a_1,\dots,a_r|k}$ satisfy properties (1)–(4) above, and also the formula $N_{a_1,\dots,a_r|k} \circ i_{K|k} = [K : k]$, by virtue of Remark 7.3.1.

Theorem 7.3.2 (Kato) *The maps $N_{a_1,\dots,a_r|k} : K_n^M(K) \to K_n^M(k)$ do not depend on the choice of the generating system $(a_1, \dots, a_r)$.*

The theorem allows us to define without ambiguity

$$N_{K|k} := N_{a_1,\dots,a_r|k} : K_n^M(K) \to K_n^M(k)$$

for all $n \geq 0$. We have the following immediate corollary:

Corollary 7.3.3 *For a k-automorphism $\sigma : K \to K$ one has $N_{K|k} \circ \sigma = N_{K|k}$.*

Proof Indeed, according to the theorem $N_{a_1,\dots,a_r|k} = N_{\sigma(a_1),\dots,\sigma(a_r)|k}$ for every system of generators $(a_1, \dots, a_r)$. □

The rest of this section will be devoted to the proof of Kato's theorem. A major step in the proof is the following reduction statement, essentially due to Bass and Tate.

Proposition 7.3.4 *Assume that Theorem 7.3.2 holds for all fields k that have no nontrivial finite extension of degree prime to p for some prime number p. Then the theorem holds for arbitrary k.*

For the proof we need some auxiliary statements.

Lemma 7.3.5 *For an algebraic extension $K|k$ the kernel of the change-of-fields map $i_{K|k} : K_n^M(k) \to K_n^M(K)$ is a torsion group. It is annihilated by the degree $[K : k]$ in the case of a finite extension.*

Proof Considering $K_n^M(K)$ as the direct limit of the groups $K_n^M(K_i)$ for all finite subextensions $k \subset K_i \subset K$ we see that it suffices to prove the second statement. Write $K = k(a_1, \dots, a_r)$ for some generators a_i. As noted above, the norm map $N_{a_1,\dots,a_r|k}$ satisfies the formula $N_{a_1,\dots,a_r|k} \circ i_{K|k} = [K : k]$, whence the claim. □

Before stating the next lemma, recall the following well-known facts from algebra (see e.g. Atiyah–Macdonald [1], Chapter 8). Given a finite field extension $K|k$ and an arbitrary field extension $L|k$, the tensor product $K \otimes_k L$ is a finite dimensional (hence Artinian) L-algebra, and as such decomposes as a finite direct sum of local L-algebras R_j in which the maximal ideal M_j is nilpotent. Let e_j be the smallest positive integer with $M_j^{e_j} = 0$. In the case when $K = k(a)$ is a simple field extension, the e_j correspond to the multiplicities of the irreducible factors in the decomposition of the minimal polynomial $f \in k[t]$ of a over L. In particular, for $K|k$ separable all the e_j are equal to 1.

Lemma 7.3.6 *In the above situation, write $K = k(a_1, \dots, a_r)$ with suitable $a_i \in K$. Denote by L_j the residue field R_j/M_j and by $p_j : L \otimes_k K \to L_j$ the natural projections. Then the diagram*

$$\begin{array}{ccc} K_n^M(K) & \xrightarrow{N_{a_1,\dots,a_r|k}} & K_n^M(k) \\ \Big\downarrow{\scriptstyle \oplus i_{L_j|K}} & & \Big\downarrow{\scriptstyle i_{L|k}} \\ \bigoplus_{j=1}^m K_n^M(L_j) & \xrightarrow{\sum e_j N_{p_j(a_1),\dots,p_j(a_r)|L}} & K_n^M(L) \end{array}$$

commutes.

Proof By the discussion above, for $r = 1$ we are in the situation of Lemma 7.2.6 and thus the statement has been already proven (modulo a

straightforward identification of the e_i with the ramification indices of the corresponding valuations on $k(t)$). We prove the general case by induction on r. Write $k(a_1) \otimes_k L \cong \oplus R_j$ for some local L-algebras R_j, and decompose the finite dimensional L-algebra $K \otimes_{k(a_1)} R_j$ as $K \otimes_{k(a_1)} R_j = \oplus R_{ij}$ for some R_{ij}. Note that $K \otimes_k L \cong \oplus_{i,j} R_{ij}$. Write L_j (resp. L_{ij}) for the residue fields of the L-algebras R_j (resp. R_{ij}), and similarly e_j and e_{ij} for the corresponding nilpotence indices. In the diagram

$$\begin{array}{ccccc} K_n^M(K) & \xrightarrow{N_{a_2,\dots,a_r|k(a_1)}} & K_n^M(k(a_1)) & \xrightarrow{N_{a_1|k}} & K_n^M(k) \\ \downarrow{\scriptstyle \oplus i_{L_{ij}|K}} & & \downarrow{\scriptstyle \oplus i_{L_j|k(a_1)}} & & \downarrow{\scriptstyle i_{L|k}} \\ \bigoplus_{i,j} K_n^M(L_{ij}) & \xrightarrow{\oplus \sum (e_{ij} e_j^{-1}) N_{\sigma_{ij}(a_2),\dots,\sigma_{ij}(a_r)|L_j}} & \bigoplus_j K_n^M(L_j) & \xrightarrow{\sum e_j N_{\sigma_j(a_1)|L}} & K_n^M(L) \end{array}$$

both squares commute by the inductive hypothesis. The lemma follows. □

Proof of Proposition 7.3.4 Write $K = k(a_1, \dots, a_r) = k(b_1, \dots, b_s)$ in two different ways. Let $\Delta \subset K_n^M(K)$ be the subgroup generated by elements of the form $N_{a_1,\dots,a_r|k}(\alpha) - N_{b_1,\dots,b_s|k}(\alpha)$ for some $\alpha \in K_n^M(K)$. Our job is to prove $\Delta = 0$. Consider the diagram of the previous lemma with $L = \bar{k}$, an algebraic closure of k. Then $L_j \cong L$ for all j and in the bottom row we have a sum of identity maps. Considering the similar diagram for $N_{b_1,\dots,b_s|k}$ we get an equality $i_{\bar{k}|k} \circ N_{a_1,\dots,a_r|k} = i_{\bar{k}|k} \circ N_{b_1,\dots,b_s|k}$, whence $\Delta \subset \ker(i_{\bar{k}|k})$. We thus conclude from Lemma 7.3.5 that Δ is a torsion group. Denoting by Δ_p its p-primary component it is therefore enough to show that $\Delta_p = 0$ for all prime numbers p. Fix a prime p, and let L be a maximal prime to p extension of k (cf. Remark 7.2.11). As all finite subextensions of $L|k$ have degree prime to p, an application of Lemma 7.3.5 shows that the restriction of $i_{L|k}$ to Δ_p is injective. On the other hand, the assumption of the proposition applies to L and hence the map $\sum e_j N_{\sigma_j(a_1),\dots,\sigma_j(a_r)|L}$ of Lemma 7.3.6 does not depend on the a_i. Therefore $i_{L|k}(\Delta_p) = 0$, which concludes the proof. □

For the rest of this section p will be a fixed prime number, and k will always denote a field having no nontrivial finite extensions of degree prime to p.

Concerning such fields, the following easy lemma will be helpful.

Lemma 7.3.7 *Let $K|k$ be a finite extension.*

1. *The field K inherits the property of having no nontrivial finite extension of degree prime to p.*
2. *If $K \neq k$, there exists a subfield $k \subset K_1 \subset K$ such that $K_1|k$ is a normal extension of degree p.*

Proof For the first statement let $L|K$ be a finite extension of degree prime to p. If $L|k$ is separable, take a Galois closure $\widetilde{L}$. By our assumption on k, the fixed field of a p-Sylow subgroup in $\mathrm{Gal}(\widetilde{L}|k)$ must equal k, so that $L = K$. If $K|k$ is purely inseparable, then $L|K$ must be separable, so $L|k$ has a subfield $L_0 \neq k$ separable over k unless $L = K$. Finally, if $K|k$ is separable but $L|K$ is not, we may assume the latter to be purely inseparable. Taking a normal closure $\widetilde{L}$, the fixed field of $\mathrm{Aut}_k(\widetilde{L})$ defines a nontrivial prime to p extension of k unless $L = K$.

The second statement is straightforward in the case when the extension $K|k$ is purely inseparable, so by replacing K with the maximal separable subextension of $K|k$ we may assume that $K|k$ is a separable extension. Consider the Galois closure $\widetilde{K}$ of K. The first statement implies that the Galois group $G := \mathrm{Gal}(\widetilde{K}|k)$ is a p-group. Now let H be a maximal subgroup of G containing $\mathrm{Gal}(\widetilde{K}|K)$. By the theory of finite p-groups (see e.g. Suzuki [1], Corollary of Theorem 1.6), it is a normal subgroup of index p in G, so we may take K_1 to be its fixed field. □

We now start the proof of Theorem 7.3.2 with the case of a degree p extension, still due to Bass and Tate.

Proposition 7.3.8 *Assume that $[K : k] = p$, and write $K = k(a)$ for some $a \in K$. The norm maps $N_{a|k} : K_n^M(k(a)) \to K_n^M(k)$ do not depend on the choice of a.*

Proof Let P be the closed point of $\mathbf{P}^1_k$ defined by the minimal polynomial of a. According to Corollary 7.2.10, the group $K_n^M(K)$ is generated by symbols of the form $\{\alpha_K, b\}$, with $\alpha \in K_{n-1}^M(k)$ and $b \in K^\times$. We compute using the projection formula for N_P (Remark 7.2.7) and Proposition 7.2.5:

$$N_{a|k}(\{\alpha_K, b\}) = N_P(\{\alpha_K, b\}) = \{\alpha, N_P(b)\} = \{\alpha, N_{K|k}(b)\}.$$

Here the right-hand side does not depend on a, as was to be shown. □

Henceforth the notation $N_{L|K} : K_n^M(L) \to K_n^M(K)$ will be legitimately used for extensions of degree p (and for those of degree 1).

Next we need the following compatibility statement with the tame symbol (which does not concern k, so there is no assumption on the fields involved). For a generalization, see Proposition 7.4.1 in the next section.

Proposition 7.3.9 *Let K be a field complete with respect to a discrete valuation v with residue field κ, and $K'|K$ a normal extension of degree p. Denote by κ' the residue field of the unique extension v' of v to K'. Then for all $n > 0$ the*

diagram

$$\begin{array}{ccc} K_n^M(K') & \xrightarrow{\partial^M_{K'}} & K^M_{n-1}(\kappa') \\ {\scriptstyle N_{K'|K}}\downarrow & & \downarrow{\scriptstyle N_{\kappa'|\kappa}} \\ K_n^M(K) & \xrightarrow{\partial^M_K} & K^M_{n-1}(\kappa) \end{array}$$

commutes.

The notes of Sridharan [1] have been helpful to us in writing up the following proof. We begin with a special case.

Lemma 7.3.10 *The compatibility of the proposition holds for symbols of the form $\alpha = \{a', a_2, \ldots, a_n\} \in K_n^M(K')$, with $a' \in K'^\times$ and $a_i \in K^\times$.*

Proof Using Lemma 7.1.2, multilinearity and graded-commutativity we may assume that $v(a_i) = 0$ for $i > 2$ and $0 \le v(a'), v(a_2) \le 1$. Setting $f := [\kappa' : \kappa]$ and denoting by e the ramification index of $v'|v$ we have the formula

$$f \cdot v' = v \circ N_{K'|K} \tag{8}$$

(see Appendix, Proposition A.6.8 (2)). Now there are four cases to consider.

Case 1 $v'(a') = v(a_2) = 0$. Then $v(N_{K'|K}(a')) = 0$, so using the projection formula we obtain $\partial^M_K(N_{K'|K}(\alpha)) = 0$, and likewise $\partial^M_{K'}(\alpha) = 0$.

Case 2 $v'(a') = 1$, $v(a_2) = 0$. In this case Remark 7.3.1 implies that with the usual notations $N_{\kappa'|\kappa}(\partial^M_{K'}(\alpha)) = f\{\bar a_2, \ldots, \bar a_n\}$. On the other hand, from (8) we infer that $N_{K'|K}(a') = u\pi^f$ for some unit u and local parameter π for v. So using the projection formula and the multilinearity of symbols we get $N_{K'|K}(\alpha) = f\{\pi, a_2, \ldots, a_n\} + \{u, a_2, \ldots, a_n\}$. This element has residue $f\{\bar a_2, \ldots, \bar a_n\}$ as well.

Case 3 $v'(a') = 0$, $v(a_2) = 1$. Then $a_2 = u'\pi'^e$ for some unit u' and local parameter π' for v', so using graded-commutativity and multilinearity of symbols we obtain $\partial^M_{K'}(\alpha) = -e\{\bar a', \bar a_3, \ldots, \bar a_n\}$. This element has norm $-e\{N_{\kappa'|\kappa}(\bar a'), \bar a_3, \ldots, \bar a_n\}$ by the projection formula. On the other hand, $\partial^M_K(N_{K'|K}(\alpha)) = -\{\overline{N_{K'|K}(a')}, \bar a_3, \ldots, \bar a_n\}$. The claim now follows from the equality $\overline{N_{K'|K}(a')} = N_{\kappa'|\kappa}(\bar a')^e$, which is easily verified in both the unramified and the totally ramified case.

Case 4 $v'(a') = v(a_2) = 1$. Write $a' = \pi'$, $a_2 = \pi$, $\pi = u'\pi'^e$, $N_{K'|K}(\pi') = u\pi^f$ as above. Then using multilinearity and Lemma 7.1.2 we get

$$\begin{aligned}\partial^M_{K'}(\alpha) &= \partial^M_{K'}(\{\pi', u', a_3, \ldots, a_n\} + e\{\pi', -1, a_3, \ldots, a_n\}) \\ &= \{(-1)^e \bar u', \bar a_3, \ldots, \bar a_n\}\end{aligned}$$

which has norm $\{(-1)^{ef} N_{\kappa'|\kappa}(\bar u'), \bar a_3, \ldots, \bar a_n\}$. On the other hand, using the projection formula we obtain as above

$$\begin{aligned}&\partial^M_K(N_{K'|K}(\alpha)) = \partial^M_K(\{u\pi^f, \pi, a_3, \ldots, a_n\}) = \\ &= \partial^M_K(-\{\pi, u, a_3, \ldots, a_n\} + f\{\pi, -1, a_3, \ldots, a_n\}) = \{(-1)^f \bar u^{-1}, \bar a_3, \ldots, \bar a_n\}.\end{aligned}$$

So it is enough to see $(-1)^{ef} N_{\kappa'|\kappa}(\bar u') = (-1)^f \bar u^{-1}$. Notice that in the above computations we are free to modify π and π' by units. In particular, in the case when $e = 1$ and $f = p$ we may take $\pi = \pi'$, so that $u' = u = 1$ and the equality is obvious. In the case $e = p$, $f = 1$ the element π' is a root of an Eisenstein polynomial $x^p + a_{p-1}x^{p-1} + \cdots + a_0$ and we may take $\pi = a_0$. Then $u = (-1)^p$ and $\bar u' = -1$, so we are done again. □

Proof of Proposition 7.3.9 Let α be an element of $K_n^M(K')$, and set $\delta := \partial^M_K(N_{K'|K}(\alpha)) - N_{\kappa'|\kappa}(\partial^M_{K'}(\alpha))$. We prove $\delta = 0$ by showing that δ is annihilated both by some power of p and by some integer prime to p.

By Corollary 7.2.10, if $K^{(p)}$ denotes a maximal prime to p extension of K, the image of α in $K_n^M(K'K^{(p)})$ is a sum of symbols of the shape as in Lemma 7.3.10 above (for the extension $K'K^{(p)}|K^{(p)}$). These symbols are all defined at a finite level, so the lemma enables us to find some extension $L|K$ of degree prime to p so that

$$\delta^L := \partial^M_L(N_{LK'|L}(i_{LK'|K'}(\alpha))) - N_{\kappa'_L|\kappa_L}(\partial^M_{LK'}(i_{LK'|K'}(\alpha))) = 0.$$

Now since $K'|K$ has degree p, we have $LK' \cong L \otimes_K K'$. This implies that the valuations $v'|v$ and their unique extensions $v'_L|v_L$ have the same ramification index e, and hence by Remark 7.1.6 (2) the tame symbol $\partial^M_{LK'}$ is the e-th multiple of ∂^M_L on symbols coming from $K_n^M(L)$, just like the tame symbol $\partial^M_{K'}$ is the e-th multiple of ∂^M_K on $i_{K'|K}(K_n^M(K))$. On the other hand, by Lemma 7.3.6 the norm map $N_{LK'|L}$ is the base change of $N_{K'|K}$ to $K_n^M(LK')$. It follows from these remarks that we have $i_{L|K}(\delta) = \delta^L$, and hence $i_{L|K}(\delta) = 0$. Thus $[L : K]\delta = 0$ by Lemma 7.3.5.

To see that δ is annihilated by some power of p, we look at the base change $K' \otimes_K K'$. Assume first that $K'|K$ is separable. Then it is Galois by assumption, so $K' \otimes_K K'$ splits as a product of p copies of K'. Therefore it is obvious that the required compatibility holds for α after base change to K'. But now there is a difference between the unramified and the ramified case. In the unramified case the residue fields in the copies of K' all equal κ, so the compatibilities of Remark 7.1.6 (2) and Lemma 7.3.6 apply with all ramification indices equal to 1, and we conclude as above that $i_{K'|K}(\delta) = 0$, hence $p\delta = 0$. In the ramified case the said compatibilities apply with ramification indices equal to p on the level of residue fields, so we conclude $pi_{K'|K}(\delta) = 0$ and $p^2\delta = 0$. Finally, in the case when $K'|K$ is purely inseparable, the tensor product $K' \otimes_K K'$ is a *eory*

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local ring with residue field κ' and nilpotent maximal ideal of length p. After base change to K' we therefore arrive at a diagram where both norm maps are identity maps, so the required compatibility is a tautology. Remark 7.1.6 (2) and Lemma 7.3.6 again apply with ramification indices equal to p, so we conclude as in the previous case that $p^2\delta = 0$. □

Corollary 7.3.11 *Let $L|k$ be a normal extension of degree p, and let P be a closed point of the projective line $\mathbf{P}^1_k$. Then the diagram*

$$\begin{array}{ccc} K_n^M(L(t)) & \xrightarrow{\oplus \partial_Q} & \bigoplus_{Q\mapsto P} K_{n-1}^M(\kappa(Q)) \\ \downarrow{\scriptstyle N_{L(t)|k(t)}} & & \downarrow{\scriptstyle \Sigma N_{\kappa(Q)|\kappa(P)}} \\ K_n^M(k(t)) & \xrightarrow{\partial_P} & K_{n-1}^M(\kappa(P)) \end{array}$$

commutes for all $n > 0$.

Proof Denote by $\widehat{K}_P$ (resp. $\widehat{L}_Q$) the completions of $k(t)$ (resp. $L(t)$) with respect to the valuations defined by P and Q. In the diagram

$$\begin{array}{ccccc} K_n^M(L(t)) & \longrightarrow & \bigoplus_{Q\mapsto P} K_n^M(\widehat{L}_Q) & \xrightarrow{\oplus \partial_Q} & \bigoplus_{Q\mapsto P} K_{n-1}^M(\kappa(Q)) \\ \downarrow{\scriptstyle N_{L(t)|k(t)}} & & \downarrow{\scriptstyle \Sigma N_{\widehat{L}_Q|\widehat{K}_P}} & & \downarrow{\scriptstyle \Sigma N_{\kappa(Q)|\kappa(P)}} \\ K_n^M(k(t)) & \longrightarrow & K_n^M(\widehat{K}_P) & \xrightarrow{\partial_P} & K_{n-1}^M(\kappa(P)) \end{array} \quad (9)$$

the right square commutes by the above proposition. Commutativity of the left square follows from Lemma 7.2.6 (or Lemma 7.3.6), noting that $L(t)\otimes_{k(t)}\widehat{K}_P$ is a direct product of fields according to Proposition A.6.4 (1) of the Appendix (and the remark following it). The corollary follows. □

Now comes the crucial step in the proof of Theorem 7.3.2.

Lemma 7.3.12 *Let $L|k$ be a normal extension of degree p, and let $k(a)|k$ be a simple finite field extension. Assume that L and $k(a)$ are both subfields of some algebraic extension of k, and denote by $L(a)$ their composite. Then for all $n \geq 0$ the diagram*

$$\begin{array}{ccc} K_n^M(L(a)) & \xrightarrow{N_{a|L}} & K_n^M(L) \\ \downarrow{\scriptstyle N_{L(a)|k(a)}} & & \downarrow{\scriptstyle N_{L|k}} \\ K_n^M(k(a)) & \xrightarrow{N_{a|k}} & K_n^M(k) \end{array}$$

commutes.

Proof Let P (resp. Q_0) be the closed point of $\mathbf{P}^1_k$ (resp. $\mathbf{P}^1_L$) defined by the minimal polynomial of a over k (resp. L). Given $\alpha \in K_n^M(L(a))$, we have $N_{a|L}(\alpha) = -\partial_\infty^M(\beta)$ for some $\beta \in K_{n+1}^M(L(t))$ satisfying $\partial_{Q_0}^M(\beta) = \alpha$ and $\partial_Q^M(\beta) = 0$ for $Q \neq Q_0$. Corollary 7.3.11 yields

$$\partial_P^M(N_{L(t)|k(t)}(\beta)) = \sum_{Q\mapsto P} N_{\kappa(Q)|\kappa(P)}(\partial_Q^M(\beta)) = N_{\kappa(Q_0)|\kappa(P)}(\alpha),$$

and, by a similar argument, $\partial_{P'}^M(N_{L(t)|k(t)}(\beta)) = 0$ for $P \neq P'$. Hence by definition of $N_{a|k}$ we get

$$N_{a|k}(N_{L(a)|k(a)}(\alpha)) = -\partial_\infty^M(N_{L(t)|k(t)}(\beta)).$$

On the other hand, since the only point of $\mathbf{P}^1_L$ above ∞ is ∞, another application of Corollary 7.3.11 gives

$$\partial_\infty^M(N_{L(t)|k(t)}(\beta)) = N_{L|k}(\partial_\infty^M(\beta)).$$

Hence finally

$$N_{a|k}(N_{L(a)|k(a)}(\alpha)) = -N_{L|k}(\partial_\infty^M(\beta)) = N_{L|k}(N_{a|L}(\alpha)).$$

□

At last, we come to:

Proof of Theorem 7.3.2 As noted before, it is enough to treat the case when k has no nontrivial extension of degree prime to p for a fixed prime p. Let p^m be the degree of the extension $K|k$. We use induction on m, the case $m = 1$ being Proposition 7.3.8. Write $K = k(a_1,\dots,a_r) = k(b_1,\dots,b_s)$ in two different ways. By Lemma 7.3.7 (2) the extension $k(a_1)|k$ contains a normal subfield $k(\bar a_1)$ of degree p over k. Applying Lemma 7.3.12 with $a = a_1$ and $L = k(\bar a_1)$ yields $N_{a_1|k} = N_{\bar a_1|k}\circ N_{a_1|k(\bar a_1)}$. So by inserting $\bar a_1$ in the system of the a_i and reindexing we may assume that $[k(a_1):k] = p$, and similarly $[k(b_1):k] = p$. Write K_0 for the composite of $k(a_1)$ and $k(b_1)$ in K, and choose elements c_i with $K = K_0(c_1,\dots,c_t)$. Note that by Lemma 7.3.7 (1) the fields $k(a_1)$ and $k(b_1)$ have no nontrivial prime to p extensions, so we may apply induction to conclude that

$$N_{a_2,\dots,a_r|k(a_1)} = N_{K_0|k(a_1)}\circ N_{c_1,\dots,c_t|K_0} \quad\text{and}\quad N_{b_2,\dots,b_s|k(b_1)} = N_{K_0|k(b_1)}\circ N_{c_1,\dots,c_t|K_0}.$$

On the other hand, Lemma 7.3.12 for $a = a_1$ and $L = k(b_1)$ implies

$$N_{a_1|k}\circ N_{K_0|k(a_1)} = N_{b_1|k}\circ N_{K_0|k(b_1)}.$$

The above equalities imply $N_{a_1,\dots,a_r|k} = N_{b_1,\dots,b_s|k}$, as desired. □