Triangulated Witt Groups

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Algebra Seminar, KU

September 19, 2013

Definition

Main references used are given below.

- Suppose $\mathcal{A}$ is an abalian category.
- A subcategory $\mathcal{E}$ of $\mathcal{A}$ is called an exact category, if for any exact sequence

$$0 \longrightarrow P \longrightarrow M \longrightarrow Q \longrightarrow 0 \quad in \quad \mathcal{A},$$

$$P, Q \in \mathcal{E} \Longrightarrow M \in \mathcal{E}.$$

- Let A be a commutative ring. The category $\mathcal{P}(A)$ of all fintely generated projective $A-$module is an exact category.

Duality

Suppose $\mathcal{E}$ is an exact category and $* : \mathcal{E} \longrightarrow \mathcal{E}$ is a contravarient functor. We denote $*(M) = M^*$. We say $*$ is a duality (involotion) on $\mathcal{E}$, if

- there is a natural equevalence (isomorphism)

$$\pi : Id \stackrel{\sim}{\to} *o* \quad \ni \forall \; objects \; M \in \mathcal{E} \quad Id_{M^*} = (\pi_M)^* \pi_{M^*}.$$

- This means, $\forall$ *objects* $M \in \mathcal{E}$ $\exists$, "natural isomorphisms"

$$\pi_M : M \stackrel{\sim}{\to} M^{**} \quad \ni \quad \begin{array}{ccc} M^* & \xrightarrow{\pi_{M^*}} & M^{***} \\ & \searrow\!\!\!\!\!= & \downarrow \pi_{M^*} \\ & & M^* \end{array} \quad commute.$$

Duality

- We say, $(\mathcal{E}, *, \pi)$ is an exact category with duality.
- Example. For a projective $A-$module P, let

$$P^* = Hom(P, A) \quad \textit{and} \quad \pi : P \xrightarrow{\sim} P^{**} \quad \textit{be the evalaution.}$$

Then $(\mathcal{P}(A), *, \pi)$ is an exact category with duality.

Symmetric spaces

Suppose $(\mathcal{E}, *, \pi)$ is an exact category with duality.

- For an object $P \in \mathcal{E}$, an isomorphism $\varphi : P \xrightarrow{\sim} P^*$ is called an symmetric isomorphism, if

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P^* \\ & \searrow^{\pi} & \downarrow{\varphi^*} \\ & & P^{**} \end{array} \qquad \textit{commutes}.$$

- If $\varphi : P \xrightarrow{\sim} P^*$ is a symmetric isomorphism, we say (P, φ) is a symmetric space (or symmetric form or a form).

Orthogonal Sum

Let (P, φ) and (Q, ψ) be two symmetric spaces. The orthogonal sum $\perp$ of these two forms is defined to be

$$(P, \varphi) \perp (Q, \psi) := \left(P \oplus Q, \begin{pmatrix} \varphi & 0 \\ 0 & \psi \end{pmatrix} \right)$$

Isometry

Let (P, φ) and (Q, ψ) be two symmetric spaces.

- An isomorphism $h : P \stackrel{\sim}{\to} Q$ is said to be isometry, if

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P^* \\ {\scriptstyle h}\downarrow & & \uparrow{\scriptstyle h^*} \\ Q & \xrightarrow[\psi]{} & Q^* \end{array} \qquad \textit{commutes.}$$

- Isometry is an equivalence relation.

The Witt monoid

Suppose $(\mathcal{E}, *, \pi)$ is an exact category (small) with duality.

- Let $MW(\mathcal{E})$ be the set of all isometry classes.
- Then,

$$(MW(\mathcal{E}), \perp) \quad \textit{has a monoid structure.}$$

- $(MW(\mathcal{E}), \perp)$ is said to be the Witt monoid of $(\mathcal{E}, *, \pi)$.

Lagrangian, Metabolic, Neutral

Suppose $(\mathcal{E}, *, \pi)$ is an exact category (small) with duality.

- Let (P, φ) be a symmetric space. A lagrangian of (P, φ) is a pair (L, α) such that (why he talks about admissible?)

$$0 \longrightarrow L \xrightarrow{\alpha} P \xrightarrow{\alpha^*\varphi} L^* \longrightarrow 0 \quad \textit{is exact}.$$

- The following diagram is helpful:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \xrightarrow{\alpha} & P & \xrightarrow{\alpha^*\varphi} & L^* & \longrightarrow & 0 \\ & & & & \downarrow \varphi & & \| & & \\ 0 & \longrightarrow & L & \longrightarrow & P^* & \xrightarrow{\alpha^*} & L^* & \longrightarrow & 0 \end{array}$$

Continued: Lagrangian, Metabolic, Neutral

- A symmetric space (P, φ) is said to be metabolic or neutral, if it has a lagrangian.
- Let $NW(\mathcal{E})$ be the set of all isometry classes of metabolic spaces. Then, $NW(\mathcal{E}) \subseteq MW(\mathcal{E})$ is a submonoid.

Quotient by submonoid

Let M be a monoid and N be a submonoid. We waht to define the quotient.

- For $x, y \in M$ define
$$x \sim y \quad \textit{if} \quad x + n_1 = y + n_2 \quad \textit{for some} \quad n_i, n_2 \in N.$$
This is an equivalence relation.
- Let M/N be the set of all equivalence classes $\overline{x}$ of elements of $x \in M$.
- M/N has a well define monoid structure,given by
$$\overline{x} + \overline{y} := \overline{x + y} \qquad \forall\ x, y \in M$$

The Group Structure: Quotient by submonoid

Lemma. Let $N \subseteq M$ be as above.

- Assume,

$$\forall\, x \in M \quad \exists \quad y \in M \quad \ni \quad x + y \in N.$$

Then, M/N has a group structure.

Hyperbolic Spaces

- Suppose (P, φ) is a symmetric space. Define the Hyperbolic space

$$\mathcal{H}(P) := ((P, \varphi) \perp (P, -\varphi))$$

- It follows

$$\mathcal{H}(P) \cong \left(P \oplus P^* \begin{pmatrix} 0 & 1_{P^*} \\ \pi_P & 0 \end{pmatrix}\right) \equiv \left(P \oplus P^* \begin{pmatrix} 0 & 1_{P^*} \\ 1_P & 0 \end{pmatrix}\right)$$

are isometric. In the last equality "$\equiv$" we treat $\pi_P = 1_P$.

Continued: Hyperbolic Spaces

The isometry is the top line of the commutative diagram

$$\begin{array}{ccc}
P \oplus P & \xrightarrow{\begin{pmatrix} .5 & .5 \\ \varphi & -\varphi \end{pmatrix}} & P \oplus P^* \\
{\scriptstyle (\varphi,-\varphi)} \downarrow & & \downarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
P^* \oplus P^* & \xleftarrow[\begin{pmatrix} .5 & \varphi \\ .5 & -\varphi \end{pmatrix}]{} & P^* \oplus P
\end{array}$$

Continued: Hyperbolic Spaces

Lemma: Let (P, φ) be a symmetric space. Then, $\mathcal{H}(P)$ is neutral.

Proof. Take $\alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix} : P \longrightarrow P \oplus P$ Then,

$$\alpha^* \begin{pmatrix} \varphi & 0 \\ 0 & -\varphi \end{pmatrix} = (\varphi, -\varphi) : P \oplus P \longrightarrow P^*.$$

It follows

$$0 \longrightarrow P \xrightarrow{\alpha} P \oplus P \longrightarrow P^* \longrightarrow 0 \quad \textit{is exact}.$$

The Witt Group

Let $(\mathcal{E}, *, \pi)$ be an exact category with duality. Deifine

$$W(\mathcal{E}) := W(\mathcal{E}, *, \pi) := \frac{MW(\mathcal{E})}{NW(\mathcal{E})}$$

- $W(\mathcal{E})$ has a group structure.
- $W(\mathcal{E})$ is called the Witt Group of $\mathcal{E}$, or of $(\mathcal{E}, *, \pi)$.

The Witt Group of Projective Modules

Let A be a commutative noetherian ring. We defined, the exact category $(\mathcal{P}(A), *, \pi)$ of projective modules. Define the Witt group of A as

$$W(A) := W(\mathcal{P}(A), *, \pi).$$

The Witt Group of Modules of FPDFL

Let A be a Cohen-Macaulay ring. Assume $\dim A_m = d$ for all maximal ideals.

- Let $\mathcal{A} = FPDFL(A)$ be the category of modules of finite length and finite projective dimension.
- Then, $\mathcal{A}$ is an exact category.
- For objects $M \in \mathcal{A}$, deifne $M^\vee := Ext^d(M, A)$.
- There is a natural isomorphism $\pi : M \xrightarrow{\sim} M^{\vee\vee}$.
- $(\mathcal{A}, \vee, \pi)$ defines an exact category with duality.
- The Witt Group $W(\mathcal{A}) := W(\mathcal{A}, \vee, \pi)$ is called the Witt group of FPDFL modules.

Exercise

Let F be a field.

- Let (V, φ) be a regular symmetric space over F. Then, if (V, φ) is neutral, then it is hyperbolic, in the sense of the book of Lam ([Lam]).
- Temporatily, let $\mathcal{W}(F)$ denote the Witt group of F, as defined in the book of Lam ([Lam]) and $W(F)$ denote the Witt group defined as above. Prove $\mathcal{W}(F) \stackrel{\sim}{\to} W(F)$.

$\delta-$exact Functors

K be a Δed category and $T : K \stackrel{\sim}{\to} K$ be the translation. Suppose $\delta = \pm 1$. An additive contravarient functor $\# : K \longrightarrow K$ is called $\delta-$exact,

- if $To\# = \#oT^{-1}$ (equivalently, if $\#T = T^{-1}\#$).
- and if $\forall$ *exact* Δs,

$$A \xrightarrow{u} B \xrightarrow{v} C \xrightarrow{w} TA$$

the "Dual" Δ,

$$C^{\#} \xrightarrow{v^{\#}} B^{\#} \xrightarrow{v^{\#}} A^{\#} \xrightarrow{\delta T(w^{\#})} T(C^{\#}) \quad \textit{is exact}.$$

Duality on Δed categories

A $\delta-$exact functor $\# : K \longrightarrow K$ is said to be a $\delta-$duality on K, if $\exists$ natural equivalace $\varpi : Id \stackrel{\sim}{\to} \#\# \ni \ \forall$ *objects* M in K

- The diagram

$$\begin{array}{ccc} M^{\#} & \xrightarrow{\varpi_{M^{\#}}} & M^{\#\#\#} \\ \| & \swarrow_{(\varpi_M)^{\#}} & \\ M^{\#} & & \end{array} \quad \textit{commutes}.$$

- And

$$\varpi_{T(M)} = T(\varpi_M). \quad \textit{Diagramatically}, \quad \begin{array}{ccc} TM & \xrightarrow{T(\varpi_M)} & T(M^{\#\#}) \\ \| & & \| \\ TM & \xrightarrow[\varpi_{TM}]{} & (TM)^{\#\#} \end{array}$$

Duality on Δed categories

- A triangualted category K with such a duality $\#$ is called a triangualted category with $\delta-$duality. Sometimes we denote it by (K, T, δ, ϖ) or simply by K.
- When $\delta = -1$, it is also referred to a skew duality. We are, in this case, thinking of the skew symmetric matrices.

Example I

Let A be a noetherian commutative ring with $\dim A = d$.

- The Derived category $D^b(\mathcal{P}(A))$ is a Δed category with $\delta = \pm 1$ duality, induced by $Hom(-, A)$, meaning the dual of of the first line in the textcolorredsecond line:

$$\cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow P_{-1} \longrightarrow \cdots$$

$$\begin{array}{ccccccc} \cdots \longrightarrow & P_{-1}^* & \longrightarrow & P_0^* & \longrightarrow & P_1^* & \longrightarrow \cdots \\ & \uparrow & & \uparrow & & \uparrow & \\ degreee = & 1 & & 0 & & -1 & \end{array}$$

- Same is true if we replace A by a noetherian scheme A.

Example II

Assume A ic Cohen-Macaulay. Let $\mathcal{A} = FPDFL(A)$.

- The Derived category $D^b(\mathcal{A})$ is a Δed category with $\delta = \pm 1$ duality, $\#$ is induced by $M^\vee := Ext^d(M, A)$.
- Remark. In $D^b(\mathcal{P}(A))$ and $D^b(\mathcal{A})$, for the transaltion $T(P_\bullet)$, it is customary to change the sign of the differential.
- Exercise. $K^b(\mathcal{P}(A)) \xrightarrow{\sim} D^b(\mathcal{P}(A))$.

Preview

As we saw in the case of exact category with duality, to define Witt groups of a category, we need two things:

- A concept of duality.
- A concept of Orthogonal Sum.
- A concept of Neutral.

We will define Witt groups of Δed categories with duality.

Symmetric Spaces

$(K, T, \#, \varpi)$ will denote a Δed category with $\delta-$duality.

- A symmetric space in K is a pair (P, φ) such that

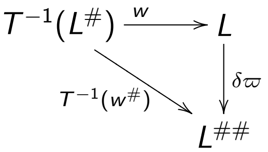
  $$\varphi : P \xrightarrow{\sim} P^{\#} \quad \textit{is isomorphism, and} \quad \begin{array}{ccc} P & \xrightarrow[\sim]{\varphi} & P^{\#} \\ {\scriptstyle \varpi}\downarrow\wr & \nearrow_{\varphi^{\#}} & \\ P^{\#\#} & & \end{array} \quad \textit{commutes}.$$

  We sometimes (often) say $\varphi = \varphi^{\#}$.
- Symmetric spaces are also referred to as symmetric form.
- (P, φ) is called skew symmetric space, if $\varphi = -\varphi^{\#}\varpi$.

Neutral Forms

$(K, T, \#, \varpi)$ be a Δed category with $\delta-$duality. A symmetric form (P, ϖ) is said to be a neutral form, if $\exists\ L, \alpha, w$ such that

- $w : T^{-1}(L^{\#}) \longrightarrow L$, $\alpha : L \longrightarrow P$ are a morphisms.
- $T^{-1}(w^{\#}) = (\delta\varpi_L)^{-1} o w$. Diagramatically,



Often written as $T^{-1}(w^{\#}) = \delta w$.

- And $(P, \varphi) = cone(w)$, which means that the triangle

$$T^{-1}(L^{\#}) \xrightarrow{w} L \xrightarrow{\alpha} P \xrightarrow{\alpha^{\#}\varphi} L^{\#} \quad \textit{is exact}.$$

The Witt Groups

$(K, T, \#, \varpi)$ be a Δed category with $\delta-$duality.

- $MW(K :) = MW(K, T, \#, \varpi)$ be the set of all isometry classes of symmetric spaces in K. The orthogonal sum gives a monoid structure on $MW(K)$. We call it the Witt monoid of K or of $(K, T, \#, \varpi)$.
- Let $NW(K) := NW(K, T, \#, \varpi) \subseteq MW(K)$ denote the submonoid of neutal spaces.
- Define,

$$W(K) := W(K, T, \#, \varpi) := \frac{MW(K, T, \#, \varpi)}{NW(K, T, \#, \varpi)}$$

A priory, $W(K)$ is monoid.

Conitnued: The Witt Groups

- Given a symmetric space (L, φ), define the Hyperbolic space $\mathcal{H}(L, \chi) := (L, \varphi) \perp (L, -\chi)$
- Lemma. $\mathcal{H}(L, \varphi)$ is a neutral form.
- Theorem. $W(K)$ is a group.
- Proof. We only need to prove the lemma. Write $P = L \oplus L$ and

$$\varphi = (\chi \perp \chi) = \begin{pmatrix} \chi & 0 \\ 0 & -\chi \end{pmatrix}, \quad \alpha = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Proof: Hyperbolic is Neutral

$$\begin{array}{ccccccc}
T^{-1}(L^\#) & \xrightarrow{w=0} & L & \xrightarrow{\alpha} & L\oplus L & \xrightarrow{\alpha^\#\varphi} & L^\# \\
\| & & \downarrow\chi & & \downarrow\varphi & & \| \\
T^{-1}(L^\#) & \xrightarrow{0} & L & \xrightarrow{\begin{pmatrix}1\\-1\end{pmatrix}} & L\oplus L & \xrightarrow{\begin{pmatrix}1 & 1\end{pmatrix}} & L^\# \\
\| & & \| & & \downarrow \begin{pmatrix}1&0\\1&1\end{pmatrix} & & \| \\
T^{-1}(L^\#) & \xrightarrow[0]{} & L^\# & \xrightarrow[\begin{pmatrix}1\\0\end{pmatrix}]{} & L^\#\oplus L^\# & \xrightarrow[\begin{pmatrix}0 & 1\end{pmatrix}]{} & L^\#
\end{array}$$

The first ∆ is exact, because the last ∆ is. The latter follows from the fact that direct sum of exact ∆s is exact. ■

The Shifted Sructure

$(K, T, \#, \varpi)$ be a Δed category with $\delta-$duality.

- Then, $T(K, T, \#, \varpi) := T(K, T, T\#, -\delta\varpi)$ is also Δed category with $-\delta-$duality.
- Likewise, $T^{-1}(K, T, \#, \varpi) := T(K, T, T^{-1}\#, \delta\varpi)$ is also Δed category with $-\delta-$duality.
- Inductively, $T^n(K, T, \#, \varpi)$ are defiend $\forall\ n \in \mathbb{Z}$. These are referred to as shifted structure.
- It is easy to see $T^2 : T^n(K, T, \#, \varpi) \xrightarrow{\sim} T^{n+4}(K, T, \#, \varpi)$ is an equivalence of categories.

Shifted Witt Groups

- Define the shifted Witt groups
$$W^n(K) := W(T^n(K, T, \#, \varpi).$$
- It follows $W^n(K) \stackrel{\sim}{\to} W^{n+4}(K)$.

Shifted Derived Categories

Suppose A is a noetherian commutative ring with $\dim A = d$. Let $D^b_{fl}(\mathcal{P}(A))$ be the subcategory of $D^b(\mathcal{P}(A))$, consisting of complexes $P_\bullet \in D^b(\mathcal{P}(A))$ with finite length homologies.

- Then, $D^b_{fl}(\mathcal{P}(A))$ is also a Δed category.
- Hence the shifted structues $T^n D^b_{fl}(\mathcal{P}(A))\ \forall\ n \in \mathbb{Z}$ are also Δed categories.
- Of particular interest is $T^d D^b_{fl}(\mathcal{P}(A))$ and shifted Witt group $W^d(D^b_{fl}(\mathcal{P}(A))$.

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