Complete Intersections

Workshop on Projective modules and A^1-homotopy theory

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Notations 0.1. Some standard notations:

1. $X = Spec(A)$ will denote a noetherian affine scheme, with $\dim X = d$. For an A-module $\mu(M) =$ minimal number of generators of M.

1 Ideal theoretic complete intersections

Loosely speaking, ideal theoretic complete intersection often reduces to the following

For an ideal $I \subseteq A$, under what condition $\mu(I) = \mu(I/I^2)$?

Things may become more meaningful and tangible, if (a) I locally complete intersection LCI or (b) the conormal bundle I/I^2 is free, as A/I-module.

Even such an expectation is unrealistic. In practice, we ask when I is image $P \twoheadrightarrow I$ of a projective A-module P of rank $\mu(I/I^2)$?

Start with the following lemma.

Lemma 1.1 (Mohan Kumar [14]). *For ideals $I \subseteq A$,*

$$\mu(I/I^2) \leq \mu(I) \leq \mu(I/I^2) + 1.$$

Proof with a purpose: Suppose $n = \mu(I/I^2)$. Then,

$I = (f_1, f_2, \ldots, f_n) + I^2. \implies \exists\, s \in I \ni (1+s)I \subseteq (f_1, f_2, \ldots, f_n).$

It follows $I = (f_1, f_2, \ldots, f_n, s)$. ∎

Observe:

$$(1+s)s = f_1 g_1 + f_2 g + \cdots + f_n g_n \qquad \text{for some} \quad g_i \in A.$$

This sets the stage of the definition of the universal ring.

1.1 The Universal Ring

Definition 1.2 (Universal Ring ([12])). For $k = \mathbb{Z}$ or a field define

$$\mathscr{A}_n := \frac{k[X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_n, Z]}{(\sum X_iY_i - Z(1+Z))}$$
$$:= k[x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n, z]$$

1. Write $\mathscr{I} = (x_1, x_2, \ldots, x_n, z)$.
2. There is a homomorphism
$$\psi : \mathscr{A}_n \longrightarrow A \quad x_i \mapsto f_i, y_i \mapsto g_i, \ z \mapsto s, \text{with} \ \ \psi(\mathscr{I}) = I.$$
3. $\mathscr{I}/\mathscr{I}^2$ is free, with basis $x_1, x_2, \ldots, x_n$. So, $\mu(\mathscr{I}/\mathscr{I}^2) = n$.

Theorem 1.3 (Mohan Kumar, Nori ([12])). *$\mathscr{I}$ cannot be the quotient of a projective $\mathscr{A}_n$-module of rank n. In particular $\mu(\mathscr{I}) \neq n$.*

Comments on the proof. The complete proof is given in ([16]). Suppose there is a surjective map $\varphi : P \twoheadrightarrow \mathscr{I}$, where P is a rojective $\mathscr{A}_n$-module with $rank(P) = n$. Then, the Chern class $c^n([P] - n) = \pm\lambda_n \in CH^n(\mathscr{A}_n)$, where $\lambda_n = cycle(\mathscr{A}_n/\mathscr{I}) \in CH^n(\mathscr{A}_n)$. Total Chow group

$$CH(\mathscr{A}_n) = \mathbb{Z} \oplus \mathbb{Z}\lambda_n \quad \text{and} \quad K_0(\mathscr{A}_n) = \mathbb{Z} \oplus \mathbb{Z}\left[\frac{\mathscr{A}_n}{\mathscr{I}}\right].$$

Using Riemann-Roch, it was also proved that $(n-1)!|C^n(P)$. ■

This was the birth of the obstruction theory approach to complete intersection.

1.2 Mohan Kumar varieties

Mohan Kumar ([12]) constructed some interesting affine varieties.

Construction 1.4. [12] Let k be a field and p be a prime number. Fix a polynomial $f(X) \in k[X]$ with $\deg(f) = p$ such that $f(0) = a \in k^*$. This polynomial $f(X)$ will be called the seed polynomial. Let $t_r = 1 + p + \cdots + p^{r-1}$.

1. Let $F(X_0, X_1) = F_1(X_0, X_1) = X_1^p f(X_0/X_1)$.
2. Inductively define $F_n = F(F_{n-1}(X_0, \ldots, X_{n-1}), a^{t_{n-1}} X_n^{p^{n-1}})$.
3. Work with (seed) polynomials $f(X)$ so that F_n is irreducible.
4. Let $\mathscr{X}_n = (F_n \neq 0) \subseteq \mathbb{P}_k^n$. Write $\mathscr{X}_n = Spec(\mathscr{B}_n)$. Then

   $$\mathscr{B}_n = k[X_0, X_1, X_2, \ldots, X_n]_{(F_n)},$$

   These $\mathscr{X}_n$ will be called Mohan Kumar varieties.
5. Fix seed polynomial, for example $f(X) = X^p + t$ where $k = k_0(t)$ and t is trancendental over a field k_0.
6. Also define $\mathscr{Y}_n = \mathcal{X}_n \cap (F_{n-1} \neq 0)$ and $\mathscr{Z}_n = \mathcal{X}_n \cap (G \neq 0)$, where $G = F_{n-1} - a^{t_{n-1}} X_n^{p^{n-1}}$. Then,

   $$\begin{array}{ccc} \Gamma(\mathscr{X}_n) & \longrightarrow & \Gamma(\mathscr{Y}_n) \\ \downarrow & & \downarrow \\ \Gamma(\mathscr{Z}_n) & \longrightarrow & \Gamma(\mathscr{Y}_n \cap \mathscr{Z}_n) \end{array} \quad \text{is a fiber product.}$$

7. Let $y = (0, \ldots, 0, 1, 1)$. Then, $y \in \mathscr{Y}_n \setminus \mathscr{Z}_n$. Let $\mathfrak{m} \subseteq \Gamma(\mathscr{X})$ represent the maximal ideal for y.

8. $\mathfrak{m}$ is complete intersection (see [12]). Let $F = \Gamma(\mathscr{Y}_n)^n$ the free module. Take an exact sequence

$$0 \longrightarrow K \xrightarrow{\iota} F \xrightarrow{\psi} \mathfrak{m}\Gamma(\mathscr{Y}_n) \longrightarrow 0 \qquad \text{and } P := F \otimes \Gamma(\mathscr{Y}_n \cap \mathscr{Z}_n).$$

It follows P is stably free, with $rank(P) = n - 1$.

9. Assume $n = p + 1$ for any prime p.

Theorem 1.5. *Let $n = p + 1$ for any prime p. Then, P is not free.*

Proof. Write $R = \Gamma(\mathscr{Y}_n \cap \mathscr{Z}_n)$. Assume P is free and $\beta : P \xrightarrow{\sim} R^{n-1}$. Then we can complete the following

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P & \xrightarrow{\iota} & R^n & \xrightarrow{\psi} & R & \longrightarrow & 0 \\
 & & \beta \downarrow \wr & & \wr \downarrow \sigma & & \| & & \\
0 & \longrightarrow & R^{n-1} & \longrightarrow & R^n & \longrightarrow & R & \longrightarrow & 0
\end{array}$$

The second line is a trivial exact sequence.

In this case, let $Q := \mathcal{P}(\Gamma(\mathscr{Y}_n)^n, \Gamma(\mathscr{Z}_n, \sigma))$ be obtained patching, via σ. It follows, there is a surjective map $\varphi : Q \twoheadrightarrow \mathfrak{m}$. So, the Chern class $C^n(Q) = \pm[y] \in CH^n(\mathscr{X}_n)$. This leads to a contradiction. ∎

1.3 Chern Classes as Obstructions I: when $k = \overline{k}$

Subsequently, Murthy took up the program to investigate whether Chern classes would work as obstructions. The final theorem is due to Murthy.

Sometimes, it is better to work with $F^dK_0(A)$ than the Chow group $CH^d(A)$. So, we give this definition.

Definition 1.6. Let A be a commutative noetherian ring with $\dim A = d$. Define

$$F^dK_0(A) = Subgroup\left(\{[A/\mathfrak{m}] \in K_0(A) : A_\mathfrak{m} \text{ is regular, } \dim A_\mathfrak{m} = d\}\right)$$

In most cases ([7]),

$$F^dK_0(A) = \left\{\left[\frac{A}{I}\right] : I \text{ is locally complete intersection, } height(I) = d\right\}$$

Definition 1.7. Let A be a commutative noetherian ring with $\dim A = d$ and $I \subseteq A$ be an ideal. An ideal J is said to be residual to I, if

1. $I + J = A$
2. J is locally complete intersection ideal of height n.
3. $\mu(I \cap J) = n$.

Under suitable conditions, if $\mu(I/I^2) = d$, then I is residual to intersection of finitely many smooth maximal ideals.

Theorem 1.8 (Murthy [15])**.** Suppose A is a reduced affine algebra, over an algebraically closed field k, with $\dim A = d$. Let $I \subseteq A$ be an ideal such that $\mu(I/I^2) = d$ and let J be residual to I. Then, there is a surjection $P \twoheadrightarrow J$ such that

1. P is a projective A-module with $rank(P) = d$.
2. $z = [P] - d \in F^d K_0(A)$.
3. $(n-1)!z = \left[\frac{A}{J}\right]$.
4. In particular, if I is locally complete intersection, then $(n-1)!z = -\left[\frac{A}{I}\right]$.

One would desire to put this theorem in a more formal way, namely in terms of Chern classes in Chow groups. Some care is needed, when $X = Spec(A)$ is not smooth.

1. First, for a projective A-module P, with $rank(P) = d$, we can define the top Chern class
$$C_d(P) = \sum_{i=0}^{n} (-1)^i [\wedge^i P^*] \in F^d K_0(A) \subseteq K_0(A).$$
In fact,
$$C_n(P) = \left[\frac{A}{I}\right] \quad \text{whenever } \exists\, P^* \twoheadrightarrow I \text{ surjection, and } I \text{ is LCI.}$$
2. With A as in (1.8), $F^d K_0(A)$ is divisible ([15, 2.10]).
So, $CH^d(A) \twoheadrightarrow F^n K_0(A)$ is surjective.

In fact, if $\dim(Sing(Spec(A))) \leq d-2$, then $F^dK_0(A)$ is divisible and torsion free (Srinivas).

3. There is a version of Riemann-Roch in ([4], assume weak-Euler class is defined):

$$\begin{array}{ccc} F^dK_0(A) & \xrightarrow{e} & CH^d(A) \\ & \searrow^{\pm(d-1)!} & \downarrow \\ & & F^dK_0(A) \end{array} \qquad \begin{array}{ccc} CH^d(A) & \xrightarrow{e} & F^dK_0(A) \\ & \searrow^{\pm(d-1)!} & \downarrow \\ & & CH^d(A) \end{array} \quad \text{commute.}$$

Therefore, when $\dim(Sing(Spec(A))) \leq d-2$ then $CH^d(A) \xrightarrow{\sim} F^dK_0(A)$ is an isomorphism of torsion free groups. (*[15] assumes regularity, due to Srinivas and Levine.*)

4. Summary: If A is as in theorem 1.8 and if $\dim(Sing(Spec(A))) \leq d-2$, then the top Chern class $c^d(P) \in CH^d(A)$ is defined. So, theorem 1.8 can be stated in terms this Chern class.

Theorem 1.9 (Murthy [15, 10])**.** Let A be s in theorem 1.8 and $\dim(Sing(Spec(A))) \leq d-2$. Suppose P is a projective A-module with $rank(P) = d$ and I is a locally complete intersection ideal of height d. Then

$$c_n(P) = cycle(A/I) \iff \exists \text{ a surjection } P^* \twoheadrightarrow I.$$

In this case, any surjective homomorpism $\varphi : P^* \twoheadrightarrow I/I^2$ lifts to a surjection:

$$\begin{array}{ccc} P^* & \overset{f}{\dashrightarrow} & I \\ & \searrow_{\varphi} & \downarrow \\ & & I/I^2 \end{array}$$

1.4 Chern Classes as Obstructions II: Real varieties

S. M. Bhatwadekar and Raja Sridharan first considered real smooth affine varities. Some of the final results are in ([1]).

Theorem 1.10 (BDM [1]). Let $X = Spec(A)$ be a smooth affine variety over reals $\mathbb{R}$, with $\dim \mathbb{R} = d \geq 2$ and the canonical module $K = \wedge^d(\Omega_{A/\mathbb{R}})$. Let P be a projective A-module of rank d and let $\wedge^d(P) = 0$. Assume the $C^n(P) = 0 \in CH_0(X)$ Then $P \simeq A \oplus Q$ in the following cases:

1. $X(\mathbb{R})$ has no compact connected component; or n is odd.
2. For every compact connected component C of $X(\mathbb{R})$, $L_C \not\simeq K_C$ where K_C and L_C induced line bundles on C.

Moreover, if n is even and L is a rank 1 projective A-module such that there exists a compact connected component C of $X(\mathbb{R})$ with the property that $L_C \simeq K_C$, then there exists a projective A-module P of rank n such that $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ (hence $C_n(P) = 0$) but P does not have a free summand of rank 1.

The structure theorem for Euler class groups was proved, and used to prove (1.10).

Theorem 1.11 (BDM, Structure Theorem([1])). Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field $\mathbb{R}$ of real numbers and let $K = \wedge^n(\Omega_{A/\mathbb{R}})$ be the canonical module of A. Let L be a projective A-module of rank 1. Let $C_1, \cdots, C_r, C_{r+1}, \cdots, C_t$ be the compact connected components of $X(\mathbb{R})$ in the Euclidean topology. Let K_{C_i} and L_{C_i} denote the induced line bundles on C_i. Assume that

$$L_{C_i} \simeq K_{C_i} \quad for\ 1 \leq i \leq r$$
$$L_{C_i} \not\simeq K_{C_i} \quad for\ r+1 \leq i \leq t$$

Then, $E^d(\mathbb{R}(X), L) = \mathbb{Z}^r \oplus (\mathbb{Z}/(2))^{t-r}$.

Remark. There are similar ctructures for Chow groups $CH^d(X)$.

These groups coincide with the topological obstruction groups.

Theorem 1.12 (-Sheu[11]). Use the same notations as in (1.11). Let P be a projective $\mathbb{R}(X)$-module with $rank(P) = d$ and $\mathcal{E}$ be the corresponding vector bundle on $X(\mathbb{R})$. Then, ther is a natural isomorphism of groups:

$$\zeta : E^d(\mathbb{R}(X), \wedge^d P) \xrightarrow{\sim} H^n\left(X(\mathbb{R}), \mathcal{G}_{\wedge^n \mathcal{E}^*}\right) \quad \text{and} \quad \zeta(e(P)) = w_n(\mathcal{E}^*)$$

where $w_n(\mathcal{E}^*)$ is the Whitney (obstruction) class of $\mathcal{E}^*$, in the cohomology group, we coefficients in $\wedge^n \mathcal{E}^*$.

Question 1.13. A few questions:

1. Suppose A is a real smooth affine algebra over $\mathbb{R}$, as in1.10 (Good Cases) and I is an ideal.

   (a) If $\mu(I/I^2) = d$, whether there is a surjection $P \twoheadrightarrow I$, for some projective A-module with $rank(P) = d$?

   (b) If I is complete intersection ideal with $height(I) = d$ and a projective A-module P of rank d, whether $C_n(P) = cycle(A/I) \implies \exists$ surjection $P^* \twoheadrightarrow I$?

2. Is it possible to give a version of theorem (1.10) when $X = Spec(A)$ is non-smooth over $\mathbb{R}$. Murthy's paper ([15]) has good amount of results.

3. For complex smooth varieties $X = Spec(A)$, there should be version the structure theorem (1.11) and the isomorphism (1.12). Note in this case, Euler class group $E^d(A, L) \xrightarrow{\sim} CH^d(A)$.

4. ([1]) Let $X = \mathrm{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over a field k of characteristic 0. Let P be a projective A-module of rank n such that $C_n(P) = 0$ in $CH_0(X)$. Then, does there exist a projective A-module Q of rank $n-1$ such that $P \oplus A \simeq Q \oplus A \oplus A$?

2 Classical results: Ideal theoretic

I feel not much has been accomplished.

I will give a list of three classes of results:

1. On ideals I in noetherian commutative rings A with $\dim A = d$.

2. On ideals I in polynomial rings $A = k[X_1, \ldots, X_d]$ over fields k.

3. On ideals I in polynomial rings $A = R[X]$, over noetherian commutative rings R.

Lemma 2.1. Suppose A is noetherian commutative ring with $\dim A = d$ and I is an ideal with $\mu(I/I^2) = n \geq d + 1$. Then, $\mu(I) = n$.

Theorem 2.2 ([6])**.** Let $R = A[X]$ be a polynomial ring over a noetherian commutative ring A and let I be an ideal of R that contains a monic polynomial. If $\mu(I/I^2) \geq \dim(R/I) + 2$, then $\mu(I) = \mu(I/I^2)$. ($R = k[X_1, \ldots, X_d]$ *over a field* k*, this is due to Mohan Kumar* ([13]).)

Question 2.3. 1. Suppose I is an ideal of in polynomial ring $R = k[X_1, \ldots, X_d]$ over a field k. Is $\mu(I) = \mu(I/I^2)$?

3 Classical results: Set theoretic

Theorem 3.1. Suppose I is an ideal in noetherian commutative rings A with $\dim A = d$. Then, I is set theoretically generated by $d + 1$ elements.

Theorem 3.2 (Eisenbud-Evans)**.** Suppose $R = A[X]$ is a polynomial ring over a commutative noetherian ring A with $\dim A = d$. Then any ideal I of R is set theoretically generated by $d + 1$ elements.

Theorem 3.3 ([8])**.** Suppose $R = A[X]$ is a polynomial ring over a noetherian commutative ring A and I is a locally complete intersection ideal of R, with $\dim(R/I) \geq 1$. If I contains a monic polynomial, then I is set theoretically generated by d elements where $d = \dim A$.

3.1 Boratyński's Theorem

Theorem 3.4 (Boratyński)**.** Let A be a commutative ring and let I be an ideal in A such that $I = (f_1, \ldots, f_n) + I^2$. Let $J = (f_1, \ldots, f_{n-1}) + I^{(n-1)!}$. Then J is the image of a projective A-module P of rank n. It follows immediately,

If $A = k[X_1, \ldots, X_d]$, over a field k. Then, $\mu(I/I^n) = n \implies$, I is set theoretically generated by n elements.

This theorem lived to play a very central role subsequently. Follwing are consequences:

1. Mohan Kumar and Nori used it for the universal construction. Their computations show for the ideal $\mathscr{I} = (x_1, \ldots, x_n, z)$ in the universal ring $\mathscr{A}_n$ and $\mathcal{J} = (x_1, \ldots, x_{n-1}) + I^{n-1}$, he proved ([15]), there is a projective $\mathscr{A}_n$-module $\tilde{P}$ with $rank(\tilde{P}) = n$ and a surjection $\tilde{P} \twoheadrightarrow \mathcal{J}$, such that
$$[\tilde{P}] - n = -\left[\frac{\mathscr{A}_n}{\mathscr{I}}\right] \in K_0(\mathscr{A}_n)$$

2. Murthy ([15]), "dragged" $\tilde{P}$ down to a projective A-module, and made Boratyński's construction stronger, by asserting that we can assume
$$[P] - n = -\left[\frac{A}{I}\right] \in K_0(A).$$

3. I used it, in the version of the Reimann-Roch ([4]) allued to, and else where.

4. Boratyński's theorem is omnipresent in complete intersection. It may be worthwhile to point out, $(n-1)!$ is omnipresent:

   (a) It is in, Suslin's theorem on completion of unimodular rows.

   (b) In Borartyński's theorem,

   (c) In Riemann-Roch.

   These are not unrelated coincidences.

Theorem 3.5 (-Roy [9]). Let $R = A[X]$ be a polynomial ring over a commutative noetherian ring A and I be an ideal of R that contains a monic polynomial. Suppose $I = (f_1, \ldots, f_n) + I^2$ and $J = (f_1, \ldots f_{n-1}) + I^{(n-1)!}$. Then $\mu(J) = n$.

Theorem 3.6 (Ferrand-Szpiro, Mohan Kumar [13]). Let $R = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k and I be a locally complete intersection ideal of R with $height(I) = n - 1$ (a curve in $\mathbb{A}^n$). Then, $\sqrt{I} = \sqrt{(f_1, \ldots, f_{n-1})}$ for some $n - 1$ elements $f_i \in I$.

Here is a monic polynomial version.

Theorem 3.7 ([8]). Suppose $R = A[X]$ is a polynomial ring over a noetherian commutative ring A, with $d = \dim A$ and I is a locally complete intersection ideal of R, with $\dim(R/I) \geq 1$. If I contains a monic polynomial, then I is set theoretically generated by d elements where $d = \dim A$.

3.2 Cowsik-Nori's theorem on curves in n spaces

Theorem 3.8 (Cowsik-Nori [3])**.** Suppose $A = k[X_1, \ldots, X_n]$ is a polynomial ring, over a field k positive characteristic p. Let I be an ideal of pure height $n-1$. Then I is set theoretically generated by $n-1$ elements.

Same is true for the projective space curves in $\mathbb{P}^n$.

Question 3.9. 1. Suppose $A = k[X_1, \ldots, X_n]$ is a polynomial ring, over a field k. Suppose I is an ideal of pure height $n-1$, then whether or not I is set theoretically generated by $n-1$ elements?

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