Quillen K-Theory
A reclamation in Commutative Algebra

Satya Mandal
Department of Mathematics, KU

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Prelude

- When I was grad. Student K-theory used to be part of Commutative Algebra.
- After Quillen published his paper in 1972. He used too much topology, for most algebraist to be able to handle.
- For most part, Topology used was basic. Depending on the area of math, these are taught to the graduate students.

Reclamation and Opportunity

- I consolidated the background needed, in about 120 pages. Everyone knows parts of it, many not be the same parts for all. K-theory can be taught to algebra students.
- After Quillen's paper, Topologist did what they are good at. They did not answer what Algebraists envisioned. Algebra community did not provide their input.
- So, there is a gold mine of research potential in Algebra.
  - Describe these groups algebraically.
  - Propose newer questions, to simplify and naturalize these proofs

Ch. 1: Category Theory

In the K-theory literature, they put everything in the frame work of categories, and arrows (maps). Highlights:

- A proof of Snake Lemma, for abelian categories.
- Quotient categories
- Inverting arrows (Localization). Calculus of fractions.
- Definition of **Exact categories**.

Ch. 2: On Homotopy Theory

We will see, for an exact category $\mathscr{E}$,
the K-groups $K_i(\mathscr{E})$ are homotopy groups $\pi_i(-, \star)$.
Good News: I avoided Homology Theory entirely.
In about 20 pages, I summarized the background
on topology and Homotopy Theory needed.
Another 20 pages, discussed Quasifibrations (Dold-Thom),
which would not be taught in graduate courses.

Ch. 2: On Homotopy Theory

- Given an exact sequence

$$0 \longrightarrow K_\bullet \xrightarrow{g} M_\bullet \xrightarrow{f} N_\bullet \longrightarrow 0 \quad \text{of modules}$$

(*or in an abelian category*) there is a long exact sequence

$$\cdots \longrightarrow H_n(K_\bullet) \xrightarrow{g_*} H_n(M_\bullet) \xrightarrow{f_*} H_n(N)_\bullet) \longrightarrow H_{n-1}(K_\bullet) \longrightarrow$$

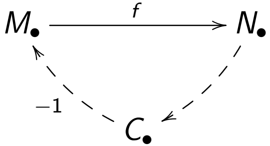
- **We can do better.**
  Given a map $f : M_\bullet \longrightarrow N_\bullet$, define the cone $C_\bullet(f)$, by

$$C_n(f) = N_n \oplus M_{n-1}$$

so that it fits in a short exact sequence

Ch. 2: On Homotopy Theory

- Or, we have a triangle (*birth of triangulated categories*)



So, we have an exact sequence

$$\cdots \longrightarrow H_n(M_\bullet) \xrightarrow{f_*} H_n(N_\bullet) \longrightarrow H_n(C_\bullet(f)) \longrightarrow H_{n-1}(M_\bullet) -$$

Ch. 2: On Homotopy Theory

- While all that comes from Topology.
  We do reverse engineering.
  Given a continuous map $f : (X, x_0) \longrightarrow (B, b_0)$,
  of pointed topological spaces,
  a topological space $F(f) := F(f, b_0)$,
  to be called **homotopy fibre**, is defined, by

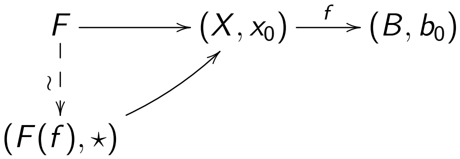
  $$F(f, b_0) = \{(x, \gamma) : x \in X, \ \gamma \text{ is a path } \quad f(x) \mapsto b_0\}$$

  Then, we have a long exact sequence

  $$\cdots \longrightarrow \pi_n(F(f), \star) \longrightarrow \pi_n(X, x_0) \longrightarrow \pi_n(B, b_0)$$

Ch. 2: On Homotopy Theory

- A **homotopy fibration** is a diagram, as follows:



where the vertical arrow is a homotopy equivalence. So, a homotopy fibration also lead to a long exact sequence.

Ch. 2: On Homotopy Theory

- Homotopy theory has a base point issue.
  Let $f : X \longrightarrow B$ be a continuous map.
  For $b \in B$, let $F_b := f^{-1}b$ be the fibre.
  We say f is a **Quasifibration**, if

  $$F_b \longrightarrow (X, x) \longrightarrow (B, b) \quad \text{is a homotopy fibration.}$$

  Consequently, $\forall b \in B$, and $x \in F_b$,
  leads to an exact sequence

  $$\cdots \longrightarrow \pi_n(F_b, x) \longrightarrow \pi_n(X, x) \longrightarrow \pi_n(B, b)$$

Ch. 2: On Homotopy Theory

- We establish (Dold-Thom) necessary and sufficient conditions for a map f to be a quasifibration, in another 20 pages.
- Two key theorems in Quillen's paper are **Theorem A, B.** In a sense, Theorem A, B are like the heart of his paper. Characterization of **Quasifibrations** becomes instrumental in the proofs.

Ch. 3. CW Complexes

In 20 pages, I consolidate the information needed about **CW complexes**, mainly from the book of Hatcher.

- A **CW complexes**, is a a topological space X together with a sequence of subspaces

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \cdots \quad \ni \quad X = \bigcup X_n$$

where X_n would be called the ***n*-skeleton**

  - X_0 is given the discrete topology, and X_n is built from X_{n-1} by attaching a family of n-cells $\mathfrak{e}_\alpha^n$ (open disk).

Ch. 3. CW Complexes

So, we have a push forward diagram

$$\begin{array}{ccc} \coprod_{\alpha\in\mathcal{I}^n}\mathbb{S}^{n-1} & \xrightarrow{\coprod\varphi_\alpha} & X_{n-1} \\ \big\downarrow & & \big\downarrow \\ \coprod_{\alpha\in\mathcal{I}^n}\mathbb{D}^{n} & \underset{\coprod\Phi_\alpha}{\dashrightarrow} & X_n \end{array} \qquad \text{in } \textbf{Top}$$

Note Φ_α maps the open disk
$\mathbb{U}^n \xrightarrow{\sim} \Phi_\alpha(\mathbb{U}^n)$ homeomorphically.
X has the weak topology. This means
$U \subseteq X$ is open $\iff$ $U \cap X_n$ is open in X_n.

Ch. 3. CW Complexes

CW complexes are very natural objects.
They enjoy many natural properties, like $(\mathbb{D}^n, \mathbb{S}^{n-1})$.

- **Theorem:** Suppose (X, A) be a CW pair. Then, (X, A) has homotopy extension property (**HEP**).

Ch. 3. CW Complexes

Weak equivalences are defined in many categories.

- **Definition:** A continuous map $f : X \longrightarrow Y$ is called a **weak equivalence** if the induced maps $f_* : \pi_n(X, x) \xrightarrow{\sim} \pi_n(Y, (f(x))$ are isomorphisms $\forall n$.
- You may recall, a map of complexes of modules is called a weak equivalence, if it induces isomorphism of homologies.

Ch. 3. CW Complexes

- Among the **most frequently used results** are the following theorem of JHC Whitehead:
- **Theorem:** Let $f : X \longrightarrow Y$ be a continuous map of CW complexes. Then, f is homotopy equivalence $\Longleftrightarrow$ f is a weak equivalence.
- **Theorem:** Suppose $X = \bigcup X_n$ is a CW complex, and $x \in X_r$. Then,

$$\begin{cases} \pi_k(X_r, x) \stackrel{\sim}{\longrightarrow} \pi_k(X, x) & \text{is isomorphism} \quad \forall k \leq r-1 \\ \pi_r(X_r, x) \twoheadrightarrow \pi_r(X, x) & \text{is surjective} \end{cases}$$

Ch. 4. Simplicial Sets

The geometry of simplicial sets, further breaks down the topological information combinatorially.
For us information flow is as follows:

$$\textbf{Topology} \Longleftrightarrow \textbf{combinatorial Geometry} \Longleftrightarrow \textbf{Algebra}$$

Recall the Δ-category.
Objects in Δ are sets $[n] := \{0, 1, 2, \ldots, n\}$.
Arrows $[m] \longrightarrow [n]$ are non decreasing maps.
Such arrows are compositions of

$$\begin{cases} d^i : [n-1] \longrightarrow [n] & \textit{face} \\ s^i : [n] \longrightarrow [n-1] & \textit{degeneracies} \end{cases}$$

Ch. 4. Simplicial Sets

- Let $e_0, e_1, e_2, \ldots, \in \mathbb{R}^{\{0,1,2,\ldots\}}$ be the standard basis. Let Σ^n be the convex hull of $e_0, e_1, \ldots, e_n$. So,

$$\Sigma^n = \{(t_0, t_1, \ldots t_n) : 0 \leq t_i \leq 1, \sum t_i = 1\}$$

We say Σ^n is the **standard n-simplex**. Then

$$\Sigma^\bullet : \Delta \longrightarrow \mathbf{Top} \quad \text{is a covarint functor,}$$

also known as co-simplicial set.

Ch. 4. Simplicial Sets

- A **simplicial set** $K_\bullet$ is a contravariant functor

$$K_\bullet : \Delta \longrightarrow \textbf{Sets}$$

- The **geometric realization** of $K_\bullet$ is defined by

$$|K_\bullet| = \frac{\coprod_n K_n \times \Sigma^n}{\sim}$$

So, $\forall\ \sigma \in K_n$, there is one standard n-simples $\sigma \times \Sigma^n$.

Ch. 4. Simplicial Sets

Main thing that we need to know is the following:

- **Theorem:** Let $K_\bullet$ be a simplicial set. Then $|K_\bullet|$ is a **CW complex**.
- The classifying spaces that we define next, would be a geometric realization of a simplicial set. Hence, they would be CW complexes, and we can use everything we know about CW complexes.

Ch. 5. Classifying Spaces

Let $\mathscr{C}$ be a category (always small).

Definition: The **nerve** of a $\mathscr{C}$ is defined to be a simplical set $\mathbf{N}_\bullet(\mathscr{C})$, as follows

- An n-simple $\sigma \in \mathbf{N}_n(\mathscr{C})$ is a sequence of composable arrows

$$\sigma := X_0 \longrightarrow \cdots \longrightarrow X_r \xrightarrow{f_r} X_{r+1} \longrightarrow \cdots \longrightarrow X_n$$

- Given $\iota : [m] \longrightarrow [n]$, a map $\mathbf{N}(\iota) : \mathbf{N}_n(\mathscr{C}) \longrightarrow \mathbf{N}_m(\mathscr{C})$ is obtained by inserting identity or by composing successive arrows.

Ch. 5. Classifying Spaces

Definition: The Classifying space of $\mathscr{C}$ is defined to be

$$\mathbb{B}\mathscr{C} := |\mathbf{N}_\bullet(\mathscr{C})| \quad \text{the geometric realization.}$$

So, for any object $X \in \mathscr{C}$, we can define **homotopy groups**:

$$\pi_n(\mathscr{C}, X) := \pi_n(\mathbb{B}\mathscr{C}, X)$$

Ch. 5. Classifying Spaces

- Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be covariant functor. Then, it induces a map

$$\mathbb{B}F : \mathbb{B}\mathscr{C} \longrightarrow \mathbb{B}\mathscr{D} \quad \text{is continuous.}$$

- Further,

$$\mathbb{B} : \mathbf{Cat} \longrightarrow \mathbf{Top} \quad \text{sending} \quad \begin{cases} \mathscr{C} \mapsto \mathbb{B}\mathscr{C} \\ F \mapsto \mathbb{B}F \end{cases}$$

is a functor, where **Cat** denotes the category of all small categories and functors.

Ch. 5. Classifying Spaces

- Let $F, G : \mathscr{C} \longrightarrow \mathscr{D}$ be two functors, and $\theta : F \longrightarrow G$ be a **natural transformation**. Then, θ induces a homotopy

$$H : \mathbb{B}\mathscr{C} \times I \longrightarrow \mathbb{B}\mathscr{D} \quad \ni \quad \begin{cases} H(-,0) = \mathbb{B}F \\ H(-,1) = \mathbb{B}G \end{cases}$$

Ch. 5. Classifying Spaces

- Let Let $F : \mathscr{C} \longrightarrow \mathscr{D}$, $F : \mathscr{D} \longrightarrow \mathscr{C}$ be two functors, and assume F is **left adjoint** to G, then

$$\mathbb{B}F : \mathbb{B}\mathscr{C} \longrightarrow \mathbb{B}\mathscr{D} \qquad \text{is a } \textbf{homotopy equivalence}.$$

- Consequently, if $\mathscr{C}$ has an initial or final object, then $\mathbb{B}\mathscr{C}$ is **contractible**.
- We would be working with exact categories $\mathscr{E}$, which has a zero. So, $\mathbb{B}\mathscr{E}$ would be contractible, we would get nothing, unless we do some more work.

Ch. 5. Theorem A, B

The Plan: Given a functor $F : \mathscr{C} \longrightarrow \mathscr{D}$,
and an object $Y \in \mathscr{D}$.
Let $F^{-1}Y = \{X \in \mathscr{C} : FX = Y\}$ be the fibre.
Then, we have a sequence $F^{-1}Y \longrightarrow \mathscr{C} \xrightarrow{F} \mathscr{D}$
We would like to write down long exact sequences

$$\pi_n(F^{-1}Y, X_0) \longrightarrow \pi_n(\mathscr{C}, X_0) \longrightarrow \pi_n(\mathscr{D}, Y) \longrightarrow \pi_{n-1}(F^{-1}Y, X_0$$

In topology also, you cannot do it, without further structure.
For a scheme X and a closed subschemes $Z \subseteq X$, $U = X - Z$,
we would like to have a long exact sequences of K-groups.

Ch. 5. Theorem A, B

Definition: Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor. For $Y \in Obj(\mathscr{D})$. Define the category Y/F as follows:

$$\begin{cases} Obj\ Y/F = \{(X,u) : X \in Obj\ \mathscr{C},\ \ u : Y \longrightarrow F(X) \\ Mor_{Y/F}((X_1,u_1),(X_2,u_2)) = \{\varphi : \text{as follows}\} \end{cases}$$

the diagram,
$$\begin{array}{ccc} Y & \xrightarrow{u_1} & FX_1 \\ & {\scriptstyle u_2}\searrow & \downarrow F(\varphi) \\ & & FX_2 \end{array}$$
commutes.

In fact,, Y/F is exact analogue of **homotopy fibres**, in topology, of F, at Y.

Ch. 5. Theorem A, B

Further, given $v : Y \longrightarrow Z$, there is a functor

$$v^* : Z/F \longrightarrow Y/F \quad \text{sending} \quad (X, u) \mapsto (X, uv)$$

Dually, we can define the category F/Y.

Ch. 5. Theorem A

Theorem A: Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor.

- Assume Y/F is contractible, $\forall\ Y \in Obj(\mathscr{D})$. Then, F is a **homotopy equivalence.**
- *There is also a dual version of the theorem, by replacing Y/F by F/Y.*

Ch. 5. Theorem B

Theorem B: Let $F : \mathscr{C} \longrightarrow \mathscr{D}$ be a functor.
Assume the functors

$$v^* : Z/F \longrightarrow Y/F \qquad \text{are homotopy equivalences} \quad \forall v$$

Then, $\forall\ Y \in Obj(\mathscr{D})$, the commutative diagram

$$\begin{array}{ccc} Y/F & \xrightarrow{j} & \mathscr{C} \\ \downarrow & & \downarrow F \\ \star \cong \quad Y/1_{\mathscr{D}} & \longrightarrow & \mathscr{D} \end{array} \qquad \text{are **homotopy cartesian**} \qquad (1)$$

Ch. 5. Theorem B

So, $\forall X \in Obj(\mathscr{C})$, $FX = Y$, there is a long exact sequence:

$$\longrightarrow \pi_{n+1}(\mathscr{D}, Y) \longrightarrow \pi_n\left(Y/F, \tilde{X}\right) \longrightarrow \pi_n(\mathscr{C}, X) \longrightarrow \pi_n(\mathscr{D}, Y$$

where $\tilde{X} := (X, 1_Y) \in Obj(Y/F)$.

- *The theorem admits a dual formulation, by replacing Y/F by F/Y etc., in the diagram (1).*
- A version, replacing Y/F by the actual fibre $F^{-1}Y$ is also available.

Ch. 6. Quillen K-theory

Suppose $\mathscr{E}$ is a small exact category.
Define the category $\mathbb{Q}\mathscr{E}$ as follows.

- First $Obj\,(\mathbb{Q}\mathscr{E}) = Obj(\mathscr{E})$.
- For $X, Y \in Obj\,(\mathbb{Q}\mathscr{E})$, a morphism $X \longrightarrow Y$ in $\mathbb{Q}\mathscr{E}$, is an equivalence class of pairs (p, i) of arrows in $\mathscr{E}$, as in the diagram:

$$X \xleftarrow{p} Z \xhookrightarrow{i} Y \ \ni \exists \text{ exact seq} \begin{cases} K \hookrightarrow Z \xrightarrow{p} X \\ Z \xhookrightarrow{i} Y \twoheadrightarrow C \end{cases} \text{in } \mathscr{E}. \tag{2}$$

Ch. 6. Quillen K-theory

- In alternate jargon, p is a deflation (admissible epi), and i is an inflation (admissible mono).
- $(p, i))$, (p', i') are defined to be equivalent, if

$$\exists \text{ an isomorphism } \tau, \ \ni \quad \begin{array}{ccccc} X & \overset{p}{\twoheadleftarrow} & Z & \overset{i}{\hookrightarrow} & Y \\ \| & & \wr\downarrow\tau & & \| \\ X & \underset{p'}{\twoheadleftarrow} & Z' & \underset{i'}{\hookrightarrow} & Y \end{array} \quad \text{commutes.}$$

Such an isomorphism τ would be unique. A **morphism** $X \longrightarrow Y$ in $\mathbb{Q}\mathscr{E}$ is an equivalence class $[(p, i)]$.
A diagram, as in (2), will be denoted by (Z, p, i).

Ch. 6. Quillen K-theory

- (**Compositions**): Given two morphisms $X \longrightarrow Y$ and $Y \longrightarrow Z$, represented by $X \overset{p}{\twoheadleftarrow} W \overset{i}{\hookrightarrow} Y$, $Y \overset{q}{\twoheadleftarrow} V \overset{j}{\hookrightarrow} Z$, the composition is given by

$$\begin{array}{ccccc} & & U & \overset{i'}{\dashrightarrow} & V & \overset{j}{\hookrightarrow} & Z \\ & & {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} & & \\ X & \underset{p}{\twoheadleftarrow} & W & \underset{i}{\hookrightarrow} & Y & & \end{array} \quad \left\{ \begin{array}{l} \text{where } U = V \times_Y W \\ \text{is the pullback.} \end{array} \right. \qquad (3)$$

Ch. 6. Quillen K-theory

Example: If $\mathscr{C} = \mathscr{P}(A)$ is the category of finitely generated projective A modules, the a morphism $X \longrightarrow Y$ is a de compostion $Y = Z \oplus K \oplus C$:

$$X \twoheadleftarrow X \oplus K \hookrightarrow X \oplus K \oplus C$$

Ch. 6. Quillen K-theory

There is no natural functor from $\mathscr{E}$ to $\mathbb{Q}\mathscr{E}$. However,

$$\left\{\begin{array}{ll} \forall\ \iota : X \hookrightarrow Y \text{ inflations, associate } \iota_! := & X \xleftarrow{1_X} X \xhookrightarrow{\iota} Y \\ \forall\ p : Y \twoheadrightarrow X \text{ deflations, associate } p^! := & X \xleftarrow[p]{} Y \xhookrightarrow[1_Y]{} Y \end{array}\right.$$

Ch. 6. Quillen K-theory

Lemma: Let $\mathscr{E}$ be an exact category.
We denote the usual/classical K-groups by $K_0^c(\mathscr{E})$, etc.
Then, there is a natural isomorphism

$$K_0^c(\mathscr{E}) \xrightarrow{\sim} \pi_1(\mathbb{B}(\mathbb{Q}\mathscr{E}), 0)$$

The map is defined as follows:

- For $X \in \mathscr{E}$, there are two arrows $0 \mapsto X$ in $\mathbb{Q}\mathscr{E}$:

$$(0, 0, 0_X) = \qquad 0 \twoheadleftarrow 0 \overset{0_X}{\hookrightarrow} X$$

$$(X, 0, 1_X) = \qquad 0 \twoheadleftarrow X \overset{1_X}{\hookrightarrow} X$$

Ch. 6. Quillen K-theory

These define, two paths $0 \mapsto X$ in $\mathbb{B}(\mathbb{Q}\mathscr{E})$:

$$\gamma_0^X := \gamma(0, 0, 0_X), \quad \gamma_1^X := \gamma(X, 0, 1_X)$$

$$\text{So,} \quad \ell_X := \overline{\gamma}_0^X \gamma_1^X : \ 0 \underset{\gamma_1^X}{\overset{\gamma_0^X}{\rightrightarrows}} X \quad \text{is a loop at } 0$$

$$\text{Define,} \qquad \varphi : K_0^c(\mathscr{E}) \longrightarrow \pi_1(\mathbb{B}\mathbb{Q}\mathscr{E}, 0) \quad \text{by} \quad \varphi(X) = [\ell_X]$$

Ch. 6. Higher K-groups

Definition: Let $\mathscr{E}$ be an exact category.

$$\begin{cases} \text{Note } \pi_0\left(\mathbb{B}(\mathbb{Q}\mathscr{E}), 0\right) = 0. \quad \text{Define,} \\ K_n(\mathscr{E}) := \pi_{n+1}\left(\mathbb{B}(\mathbb{Q}\mathscr{E}), 0\right) \end{cases}$$

We can also define the ***K*-theory space**

$$\mathbf{K}\mathscr{E} = \Omega\left(\mathbb{B}(\mathbb{Q}\mathscr{E}), 0\right), \text{ the loop space. Then, } K_n(\mathscr{E}) := \pi_n\left(\mathbf{K}\mathscr{E}\right)$$

Ch. 6. Higher K-groups

- Classically, three groups $K_0^c(R)$, $K_1^c(R)$, $K_2^c(R)$, were defined. In chapter 7, we prove,

$$K_1^c(R) \cong K_1(\mathscr{P}(R)), \text{ where } R \text{ is a commutative ring.}$$

I skipped, plus construction or homology theory.

Ch. 6. Higher K-groups

Lemma Let $F : \mathscr{E} \longrightarrow \mathscr{D}$ be an exact sequence of exact functors. Then, F induces natural maps and homomorphisms

$$\begin{cases} K_n\mathscr{E} \longrightarrow K_n\mathscr{D} & \text{homomorphisms } \forall n \geq 0 \\ \mathbb{BQ}\mathscr{E} \longrightarrow \mathbb{BQ}\mathscr{D} & \text{continuous map} \\ \mathbf{K}\mathscr{E} \longrightarrow \mathbf{K}\mathscr{D} & \text{continuous map} \end{cases}$$

In a sense, these three are equivalent.

Ch. 6. Higher K-groups

Additivity Theorem: Let

$$G, F, H : \mathscr{E} \longrightarrow \mathscr{D} \quad \text{be exact functors}$$

of exact categories, such that

$$0 \longrightarrow G \longrightarrow F \longrightarrow H \longrightarrow 0 \quad \text{is also exact.} \qquad (4)$$

Then,

$$\forall\ n \geq 0 \quad F_* = G_* + H_* : K_n(\mathscr{E}) \longrightarrow K_n(\mathscr{D})$$

Ch. 6. Higher K-groups

Example: Let R be a commutative ring and $P = P_1 \oplus P_2$ be a projective R-modules. Then

$$\left\{\begin{array}{l} -\otimes P_1 \\ -\otimes P \\ -\otimes P_2 \end{array}\right. : Coh(R) \longrightarrow Coh(R) \quad \text{are exact.}$$

$$0 \longrightarrow -\otimes P_1 \longrightarrow -\otimes P \longrightarrow -\otimes P_2 \longrightarrow 0 \quad \text{is exac. So,}$$

$$(-\otimes P)_* = (-\otimes P_1)_* + (-\otimes P_2)_* : K_n(Coh(R)) \longrightarrow K_n(Coh(R))$$

Ch. 6. Higher K-groups

- In particular, there is a $K_0(R)$ **action on** $K_n(Coh(R))$:

$$K_0(R) \otimes K_n(Coh(R)) \longrightarrow K_n(Coh(R))$$

- This works for schemes X, and exact sequences

$$0 \longrightarrow P_1 \longrightarrow P \longrightarrow P_2 \longrightarrow 0 \quad \text{of locally free sheaves.}$$

Ch. 6. Higher K-groups

Resolution Theorem: Let $\mathscr{E}$ be an exact category and $\mathscr{P} \subseteq \mathscr{E}$ be a full subcategory. Assume

- For any exact sequence in $\mathscr{E}$:

$$0 \longrightarrow K \longrightarrow M \longrightarrow C \longrightarrow 0 \left\{ \begin{array}{l} K, C \in \mathscr{P} \Longrightarrow M \in \mathscr{P} \\ M, C \in \mathscr{P} \Longrightarrow K \in \mathscr{P} \end{array} \right.$$

- $\forall\ M \in Obj(\mathscr{E})$, there is a finite resolution, with P_i in $\mathscr{P}$:

$$0 \longrightarrow P_n \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow M \longrightarrow 0$$

Sometimes these are called **resolving categories.**

Ch. 6. Higher K-groups

$$\text{Then, } \begin{cases} \forall\ n \geq 0, \quad K_n(\mathscr{P}) \xrightarrow{\sim} K_n(\mathscr{E}) & \text{are isomorphisms.} \\ \mathbb{BQ}\mathscr{P} \longrightarrow \mathbb{BQ}\mathscr{E} & \text{is a homotopy equivalence.} \\ \mathbf{K}\mathscr{P} \longrightarrow \mathbf{K}\mathscr{E} & \text{is a homotopy equivalence.} \end{cases}$$

In fact, these three are equivalent statements.

Ch. 6. Higher K-groups

Example: Let R be a commutative ring. Let $\mathscr{P}(R)$ be the category of projective R-modules, and $\mathbb{H}(R)$ be the category of $M \in Coh(R)$ with finite projective dimension. Then,

$$K_n(\mathscr{P}(R)) \xrightarrow{\sim} K_n(\mathbb{H}(R))$$

This works for schemes X, and locally free sheaves.

Ch. 6. Higher K-groups

Dévissage Theorem: Let $\mathscr{A}$ be an abelian category. Let $\mathscr{B} \subseteq \mathscr{A}$ be a full subcategory, such that
$(\star)$ $\mathscr{B}$ is closed under
subobjects, quotient objects and finite product in $\mathscr{A}$.
In this case, $\mathscr{B}$ is an abelian subcategory.
Assume, every object $M \in Obj(\mathscr{A})$ has a filtration:

$$0 = M_0 \hookrightarrow M_1 \hookrightarrow \cdots \hookrightarrow M_r =: M \quad \frac{M_j}{M_{j-1}} \in Obj(\mathscr{B}) \quad \forall j.$$

Ch. 6. Higher K-groups

$$\text{Then, } \begin{cases} \forall\ n \geq 0, \quad K_n(\mathscr{B}) \xrightarrow{\sim} K_n(\mathscr{A}) & \text{are isomorphisms.} \\ \mathbb{BQ}\mathscr{B} \longrightarrow \mathbb{BQ}\mathscr{A} & \text{is a homotopy equivalence.} \\ \mathbf{K}\mathscr{B} \longrightarrow \mathbf{K}\mathscr{A} & \text{is a homotopy equivalence.} \end{cases}$$

Ch. 6. Higher K-groups

Example: Let R be a commutative ring. Let $R_{red} = \frac{R}{\sqrt{0}}$.

Then, $$K_n\left(Coh(R_{red})\right) \cong K_n(Coh(R))$$

Note, usually, $K_1\left(\mathscr{P}(R_{red})\right) \neq K_1(\mathscr{P}(R))$.
This works for schemes X.

Ch. 6. Higher K-groups

Definition: Let $\mathscr{A}$ be an abelian category. A full subctegory $\mathscr{B} \subseteq \mathscr{A}$ is defined to be a **Serre subcategory**, if, for any exact sequence in $\mathscr{A}$:

$$0 \longrightarrow K \longrightarrow M \longrightarrow C \longrightarrow 0 \qquad M \in \mathscr{B} \Longleftrightarrow K, C \in \mathscr{B}$$

In this case, the quotient category $\frac{\mathscr{A}}{\mathscr{B}}$ is defined, and

Ch. 6. Higher K-groups

Localization Theorem: Let $\mathscr{A}$ be an abelian category and $\mathscr{B} \subseteq \mathscr{A}$ be a **Serre subcategory**. Then, the sequence

$$\mathscr{B} \xrightarrow{\iota} \mathscr{A} \xrightarrow{\mathfrak{q}} \frac{\mathscr{A}}{\mathscr{B}} \qquad \text{is a homotopy fibration.}$$

Consequently, there is an exact sequence

$$\cdots \xrightarrow{\mathfrak{q}_*} K_1\left(\frac{\mathscr{A}}{\mathscr{B}}\right) \longrightarrow K_0(\mathscr{B}) \xrightarrow{\iota_*} K_0(\mathscr{A}) \xrightarrow{\mathfrak{q}_*} K_0\left(\frac{\mathscr{A}}{\mathscr{B}}\right) \longrightarrow 0$$

Ch. 6. Higher K-groups

Example: Let X be a noetherian scheme and $Z \hookrightarrow X$ be a closed subset and $U = X - Z$. Then, we have exact sequences

$$\cdots \xrightarrow{\mathfrak{q}_*} K_{n+1}\left(Coh(U)\right) \longrightarrow$$

$$K_n\left(Coh_Z(X)\right) \xrightarrow{\iota_*} K_n\left(Coh(X)\right) \xrightarrow{\mathfrak{q}_*} K_n\left(Coh(U)\right) \longrightarrow \cdots$$

$$K_n\left(Coh(Z_{red})\right) \xrightarrow{\sim} K_n\left(Coh_Z(X)\right)$$

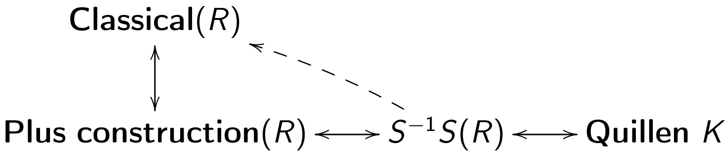
Ch. 6. Higher K-groups

Here $Coh_Z(X) \subseteq Coh(X)$ is full subcategory of objects $\mathcal{F} \in Coh(X)$, with support in Z.
The vertical isomorphism is given by Dévissage, above.

Ch. 7. Agreement

Already mentioned above $\quad K_1^c(R) \cong K_1(R)$

Consider a symmetric monoidal category $(S, \odot, \mathbf{e})$, where $\odot$ represents direct sum, and $\mathbf{e}$ the zero. There is a so called $S^{-1}S$ category. This relates to both Quillen K-theory and Plus-construction.



Ch. 8. K-Theory of rings

Now on, given a ring R (usually commutative),
$Coh(A)$ = the category of finitely generated A-modules
$\mathscr{P}(A)$ = the category of finitely generated projective A-modules.
The K-theory of $Coh(A)$ is referred to as G-theory, and

$$\begin{cases} G_n(A) := K_n(Coh(A)) \\ K_n(A) := K_n(\mathscr{P}(A)) \end{cases}$$

Ch. 8. K-Theory of rings

Main theorem in this section is the **Homotopy invariance**:

- **Theorem:** Let A be a noetherian ring and $B = A[T]$ be the polynomial ring. Then,

$$\begin{cases} \mathbb{BQ}Coh(A) \longrightarrow \mathbb{BQ}Coh(B) & \text{is a homotopy equivalence} \\ G_n(A) \xrightarrow{\sim} G_n(B) & \text{are isomorphisms} \quad n \geq 0 \end{cases}$$

- **Corollary:** Let A be a noetherian **regular** ring and $B = A[T]$ be the polynomial ring. Then,

$$K_n(A) \xrightarrow{\sim} K_n(B) \quad \text{are isomorphisms} \quad n \geq 0$$

Ch. 8. Bass-Quillen Conjecture, naturalized

Naturalized Bass-Quillen Conjecture:
Suppose A is an regular affine algebra over a (perfect) field. Let P be a $A[T]$-module, and $\overline{P} = \frac{P}{TP}$.

$$\begin{cases} \text{Is there a natural isomorphism} \quad P \xrightarrow{\sim} \overline{P} \otimes A[T]? \\ \text{Or, Is there a natural transformation} \quad P \longrightarrow \overline{P} \otimes A[T] \\ \text{in} \quad \mathbb{Q}\mathscr{P}(A[T])? \end{cases}$$

If yes, the above corollary would have more natural proof.

Ch. 9. K-Theory of schemes

Scheme theory is part of commutative algebra.

For a scheme X (usually noetherian),
$Coh(X)$ = the category of coherent $\mathcal{O}_X$-modules
$\mathscr{P}(X)$ = the category of locally free X-modules.
The K-theory of $Coh(X)$ is referred to as G-theory, and

$$\begin{cases} G_n(X) := K_n(Coh(X)) & \mathbf{G}(X) = \Omega(\mathbb{BQ}Coh(X)) \\ K_n(X) := K_n(\mathscr{P}(X)) & \mathbf{K}(X) = \Omega(\mathbb{BQ}\mathscr{P}(X)) \end{cases}$$

Two columns basically have the equivalanet information.

Ch. 9. K-Theory of schemes

The functor $\mathscr{P}(X) \longrightarrow Coh(X)$ induces maps

$$\begin{cases} K_n(X) \longrightarrow G_n(X) & \forall n \geq 0 \\ \mathbb{BQ}\mathscr{P}(X) \longrightarrow \mathbb{BQ}Coh(X) \\ \mathbf{K}(X) \longrightarrow \mathbf{G}(X) \end{cases}$$

Theorem: If X is regular and separated, then

$$\begin{cases} K_n(X) \longrightarrow G_n(X) & \text{are isomorphisms} \forall n \geq 0 \\ \mathbb{BQ}\mathscr{P}(X) \longrightarrow \mathbb{BQ}Coh(X) & \text{is homotopy equivalence} \\ \mathbf{K}(X) \longrightarrow \mathbf{G}(X) & \text{is homotopy equivalence} \end{cases}$$

Proof. Follows from resolution theorem. ■

Ch. 9. Pullback : G-Theory

Defintion: Let $f : X \longrightarrow Y$ be a map of schemes.
Then, f induces a functor

$$f^* : Coh(Y) \longrightarrow Coh(X) \quad \text{sending} \quad \mathcal{F} \mapsto f^*\mathcal{F} \qquad (5)$$

Usually, this is not exact.

Ch. 9. Pullback : G-Theory

- The restriction $f^*: \mathscr{P}(Y) \longrightarrow \mathscr{P}(X)$ is exact. So, it induces maps

$$\begin{cases} f^*: \mathbf{K}(Y) \longrightarrow \mathbf{K}(X) & \text{of } K\text{−theory spaces,} \\ f^*: K_n(Y) \longrightarrow K_n(X) & \text{of } K\text{−groups } \forall\ n \geq 0. \end{cases}$$

- If f is flat, (5) is an exact functor. So, it induces maps

$$\begin{cases} f^*: \mathbf{G}(Y) \longrightarrow \mathbf{G}(X) & \text{of } G\text{−theory spaces,} \\ f^*: G_n(Y) \longrightarrow G_n(X) & \text{of } G\text{−groups } \forall\ n \geq 0. \end{cases} \quad (6)$$

We can do better!

Ch. 9. Pullback : G-Theory

Lemma: Let $f : X \longrightarrow Y$ be a morphism of noetherian schemes. Assume Y has enough locally free sheaves and f has **finite Tor dimension**, meaning

$$\sup\left\{k : Tor_k^Y(\mathcal{F}, \mathcal{O}_X) \neq 0 \text{ for some } \mathcal{F} \in QCoh(Y)\right\} < \infty$$

Define the full subcategory of $Coh(Y)$, as follows

$$\mathfrak{C}oh(f, Y) = \left\{\mathcal{F} \in Coh(Y) : Tor_k^Y(\mathcal{F}, \mathcal{O}_X) = 0 \ \forall\ k \geq 1\right\} \quad (7)$$

Then, the restriction

$$f^* : \mathfrak{C}oh(f, Y) \longrightarrow Coh(X) \qquad \text{is an exact functor.} \quad (8)$$

Ch. 9. Pullback : G-Theory

So, this induces

$$\begin{cases} \mathbf{K}(\mathfrak{C}oh(f,Y)) \longrightarrow \mathbf{G}(X) & \text{map of } K\text{–theory spaces} \\ K_n(\mathfrak{C}oh(f,Y)) \longrightarrow G_n(X) & \text{homomorphisms of } K\text{–groups } \forall\ n \end{cases}$$

Further, every $\mathcal{F} \in Coh(Y)$ has a finite resolution by objects in $\mathfrak{C}oh(f,Y)$. By **resolution theorem**, we have

$$\begin{cases} \mathbf{K}(\mathfrak{C}oh(f,Y)) \xrightarrow{\sim} \mathbf{G}(Y) & \text{homotopy equivalence of } K\text{–theor} \\ K_n(\mathfrak{C}oh(f,Y)) \xrightarrow{\sim} G_n(Y) & \text{isomorphisms of } K\text{–groups } \forall\ n \geq \end{cases} \quad (9)$$

Combining this with (9),

Ch. 9. Pullback : G-Theory

we obtain map Pullback maps

$$f^* : \begin{cases} G_n(Y) \xleftarrow[\sim]{} K_n\left(\mathfrak{C}oh(f,Y)\right) \longrightarrow G_n(X) \\ \mathbf{G}(Y) \xleftarrow[\sim]{} \mathbf{K}\left(\mathfrak{C}oh(f,Y)\right) \longrightarrow \mathbf{G}(X) \end{cases}$$

Ch. 9. Push Forward : G-Theory

- For simplicity, consider $f : \mathrm{Spec}\,(B) \longrightarrow \mathrm{Spec}\,(A)$. Given $M \in Coh(B)$, it is not necessary that $M \in Coh(A)$. So, for $f : X \longrightarrow Y$, defining push forward

$$f_* : G_n(X) \longrightarrow G_n(Y) \quad \text{would require some work.}$$

- However, if $f : X \longrightarrow Y$ is a projective morphism, then

$$\forall \mathcal{F} \in Coh(X) \qquad \text{then,} \qquad R^k f_* \mathcal{F} \in Coh(Y) \ \ \forall k$$

where $R^k f_* \mathcal{F}$ denote the higher direct images, with $R^0 f_* \mathcal{F} = f_* \mathcal{F}$.

Ch. 9. Push Forward : G-Theory

What is a projective morphisms? We say $f : X \longrightarrow Y$ is a projective morphism, if it factors as

$$\begin{array}{ccc} X & \xrightarrow{\text{closed}} & \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} Y \\ & {\scriptstyle f}\searrow & \downarrow \\ & & Y \end{array}$$

If $Y = \mathrm{Spec}\,(A)$ it is :

$$\begin{array}{ccc} X & \xrightarrow{\text{closed}} & \mathrm{Proj}\,(A[T_0, T_1, \ldots, T_n]) \\ & {\scriptstyle f}\searrow & \downarrow \\ & & \mathrm{Spec}\,(A) \end{array}$$

Ch. 9. Push Forward : G-Theory

What is higher direct images $R^k\mathcal{F}$?

Let $f : X \longrightarrow Y$ be a morphism noetherian schemes. Given $\mathcal{F} \in QCoh(X)$, consider a injective resolution:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathscr{I}_0 \longrightarrow \mathscr{I}_1 \longrightarrow \mathscr{I}_2 \longrightarrow \cdots \quad \text{denoted by} \quad \mathscr{I}_\bullet \tag{10}$$

Apply (1) the global section $\Gamma(X, -)$ and (2) direct image functor f_* functor:

$$\begin{cases} 0 \longrightarrow \Gamma(X, \mathcal{F}) \longrightarrow \Gamma(X, \mathscr{I}_0) \longrightarrow \Gamma(X, \mathscr{I}_1) \longrightarrow \cdots \\ 0 \longrightarrow f_*\mathcal{F} \longrightarrow f_*\mathscr{I}_0 \longrightarrow f_*\mathscr{I}_1 \longrightarrow f_*\mathscr{I}_2 \longrightarrow \cdots \end{cases} \tag{11}$$

Ch. 9. Push Forward : G-Theory

For integers $k \geq 0$, define the following:
Define the **sheaf cohomology**,

$$\begin{cases} H^k(X, \mathcal{F}) = H^k(\Gamma(X, \mathscr{I}_\bullet)) & \textbf{sheaf cohomology} \\ R^k f_* \mathcal{F} = \mathcal{H}^k(f_* \mathscr{I}_\bullet) \in QCoh(Y) & \textbf{higher direct image} \end{cases} \tag{12}$$

As usual, given an exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0 \quad \text{in} \quad QCoh(X) \tag{13}$$

Ch. 9. Push Forward : G-Theory

there is a connecting morphism $\partial^k : R^k f_* \mathcal{G} \longrightarrow R^{k+1} f_* \mathcal{K}$ such that we obtain a long exact sequence

$$\cdots \longrightarrow R^k f_* \mathcal{K} \longrightarrow R^k f_* \mathcal{F} \longrightarrow R^k f_* \mathcal{G} \xrightarrow{\partial^k} R^{k+1} f_* \mathcal{K} \longrightarrow \cdots \quad (14)$$

starting at degree zero.

Ch. 9. Push Forward : G-Theory

I did the following:

- For a morphism $f : X \longrightarrow Y$, and $\mathcal{F} \in QCoh(X)$, defined $R^k f_* \mathcal{F} \in QCoh(Y)$, with $R^0 f_* \mathcal{F} = f_* \mathcal{F}$.
- I defined projective morphisms $f : X \longrightarrow Y$.

Lemma: Let $f : X \longrightarrow Y$ be a projective morphism. Then, for $\mathcal{F} \in Coh(X)$, we have
(1) $R^k f_* \mathcal{F} \in Coh(Y)$
(2) $R^k f_* \mathcal{F} = 0 \quad \forall k \gg 0$
(3) $\exists\ n_0$, such that $\forall n \geq n_0$, $R^k f_* \mathcal{F}(n) = 0$, $\forall k \geq 1$.

Ch. 9. Push Forward : G-Theory

We define push forward:

Definition: Let $f : X \longrightarrow Y$ be a projective morphism of noetherian schemes. Consider the direct image functor

$$f_* : Coh(X) \longrightarrow Coh(Y), \qquad \text{which is not necessarily exact.}$$

Consider the full subcatgory of $Coh(X)$, as follows

$$\mathfrak{C}oh(X, f) = \left\{\mathcal{F} \in Coh(X) : R^k f_* \mathcal{F} = 0 \ \forall\ k \geq 1\right\}$$

$$\text{The restriction} \qquad f_* : \mathfrak{C}oh(X, f) \longrightarrow Coh(Y), \quad \text{is exact}$$

Ch. 9. Push Forward : G-Theory

Consequently, there are maps

$$\begin{cases} \mathbf{K}\mathfrak{C}oh(X,f) \longrightarrow \mathbf{G}(Y) & \text{of } K\text{−theory spaces} \\ K_n\left(\mathfrak{C}oh(X,f)\right)) \longrightarrow G_n(Y) & \text{of the } K\text{−groups} \end{cases} \quad (15)$$

For $\mathcal{F} \in Coh(X)$ there is a finite resolution in $Coh(X)^{op}$:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \longrightarrow \cdots \longrightarrow \mathcal{F}_d \longrightarrow 0 \text{ with } \mathcal{F}_k \in \mathfrak{C}oh(X,f).$$

By resolution theorem applied to $\mathfrak{C}oh(f,X)^{op} \hookrightarrow Coh(X)^{op}$, and we have, homotopy equivalence and isomorphisms:

$$\begin{cases} \mathbf{K}\mathfrak{C}oh(X,f) = \mathbf{K}\mathfrak{C}oh(X,f)^{op} \cong \mathbf{K}(Coh(X)^{op}) = \mathbf{G}(X) \\ K_n\mathfrak{C}oh(X,f) = K_n(X,f)^{op} \cong K_n(Coh(X)^{op}) = G_n(X) \end{cases}$$

Ch. 9. Push Forward : G-Theory

Combining with (15), we have the **push forward** maps of K-theory spaces, and K-groups:

$$f_* : \begin{cases} \mathbf{G}(X) \xleftarrow{\sim} \mathbf{K}\mathfrak{C}oh(X, f) \longrightarrow \mathbf{G}(Y) \\ G_n(X) \xleftarrow{\sim} K_n\left(\mathfrak{C}oh(X, f)\right) \longrightarrow G_n(Y) \end{cases} \quad (16)$$

Ch. 9. Projection Formula: G-Theory

The following projection formula for G-theory.

Theorem: Let $f : X \longrightarrow Y$ be a projective morphism (proper) of noetherian schemes. Assume (1) f has finite Tor dimension and (2) both X, Y support ample bundles. Then,

- Recall $K_0(X)$ has an action on $G_n(X)$.
- We have,

$$f_*(x \cdot f^* y) = f_*(x) \cdot y \in G_n(Y) \qquad \forall\, x \in K_0 X,\ y \in G_n(Y)$$

Ch. 9. Projection Formula: G-Theory

So the diagram

$$\begin{array}{ccccc} K_0(X) \otimes G_n(Y) & \xrightarrow{1\times f^*} & K_0(X) \otimes G_n(X) & \xrightarrow{\cdot} & G_n(X) \\ \downarrow{\scriptstyle f_*\otimes 1} & & & & \downarrow{\scriptstyle f_*} \\ K_0(Y) \otimes G_n(Y) & & \xrightarrow{\cdot} & & G_n(Y) \end{array} \quad \text{commutes.} \tag{17}$$

Ch. 9. Projective Bundle: G-Theory

Theorem: Let Y be a noetherian scheme and let $\mathscr{E} \in \mathscr{P}(Y)$ be locally free sheaf with $rank(\mathscr{E}) = r$.
Write $\mathbb{P}\mathscr{E} = \mathrm{Proj}\,(\mathrm{Sym}(\mathscr{E}))$.
Let $f : \mathbb{P}\mathscr{E} \longrightarrow Y$ be the structure map.
Then, with notation $\zeta = [\mathcal{O}(-1)] \in K_0\,(\mathbb{P}\mathscr{E})$,
we have an isomorphism

$$\varphi_X : G_n(Y)^r \xrightarrow{\sim} G_n\,(\mathbb{P}\mathscr{E}) \quad (x_0, x_1, \ldots, x_{r-1}) \mapsto \sum_{k=0}^{r-1} \zeta^k \cdot f^* x_k \tag{18}$$

Ch. 9. Filtration of support and Gersten complex: K-Theory of schemes

Skip

Ch. 10. Projective Bundle Theorem: K-Theory

Theorem: Let Y be a noetherian scheme and let $\mathscr{E} \in \mathscr{P}(Y)$ be locally free sheaf with $rank(\mathscr{E}) = r$.
Write $\mathbb{P}\mathscr{E} = \mathrm{Proj}\left(\mathrm{Sym}(\mathscr{E})\right)$.
Let $f : \mathbb{P}\mathscr{E} \longrightarrow Y$ be the structure map.
Then, with notation $\zeta = [\mathcal{O}(-1)] \in K_0\left(\mathbb{P}\mathscr{E}\right)$, we have an isomorphism

$$\varphi_Y : K_n(X)^r \xrightarrow{\sim} K_n\left(\mathbb{P}\mathscr{E}\right) \quad (x_0, x_1, \ldots, x_{r-1}) \mapsto \sum_{k=0}^{r-1} \zeta^k \cdot f^* x_k \tag{19}$$

Ch. 10. Projective Bundle Theorem: K-Theory

- The statement of is exactly similar to the theorem on G-theory.
- The proof is much involved scheme theoretically.
- One main ingredient is construction of a **canonical resolution**, of **regular** locally free sheaves.
- Then use resolution theorem, on a tight rope walk.

Ch. 11. K-Theory of quadrics

Swan extended the projective bundle theorem to nonsingular quadric hypersurfcaes

$$Y = \mathrm{Proj}\left(\frac{R[X_0, X_1, \ldots, X_n]}{(f)}\right)$$

This is used compute K-theory of real and complex (affine) spheres $\mathbb{S}^n = (\sum_{i=0}^n X_i^2 = 1)$, by looking at the open subset

$$\mathbb{S}^n \cong (T = 1) \subseteq Y = \mathrm{Proj}\left(\frac{R[X_0, X_1, \ldots, X_n, T]}{(\sum_{i=0}^n X_i^2 - T^2)}\right)$$

Ch. 11. K-Theory of quadrics

Let $\mathfrak{q} = \sum_{1\leq i\leq j\leq n} a_{ij}X_iX_j$. Then, it relates to the bilinear form

$$B(\mathbf{X},\mathbf{Y}) = \mathbf{X}^t \begin{pmatrix} 2a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{12} & 2a_{22} & a_{23} & \cdots & a_{2n} \\ a_{13} & a_{23} & 2a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & a_{3n} & \cdots & 2a_{nn} \end{pmatrix} \mathbf{Y} \quad \text{where}$$

where $\mathbf{X}$, $\mathbf{Y}$ are column matrices. So,

$$\mathfrak{q}(\mathbf{X}) = \frac{1}{2}B(\mathbf{X},\mathbf{X})$$

Ch. 11. K-Theory of quadrics

Definition. Let R be a commutative ring, with $1/2 \in R$. By a quadratic R-module, we mean a pair (P, φ), where P is a projective R-module and $\varphi : P \longrightarrow P^*$ is a symmetric linear map. This means,

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P^* \\ \wr\downarrow & \nearrow_{\varphi^*} & \\ P^{**} & & \end{array} \quad \textit{commute.} \quad \left\{ \begin{array}{l} \text{Define} \quad \mathfrak{q} : P \longrightarrow R \\ \mathfrak{q}(x) = \frac{1}{2}\varphi(x)(x) \end{array} \right.$$

It is customary to say, $(P, \mathfrak{q})$ is a quadratic R-module.

Ch. 11. K-Theory of quadrics

- $\mathcal{S}ym(P^*) = \bigoplus_{n\geq 0} \mathcal{S}ym_n(P^*)$ be the symmetric algebra.
- Let $Quad(P)$ denote the module of all quadratic R-modules $(P, \mathfrak{q})$. Then, there is bijection

$$\mathcal{S}ym_2(P^*) \xrightarrow{\sim} Quad(P)$$

We denote the preimage of $\mathfrak{q}$ by the same notation $\mathfrak{q}$. Let

$$S(\mathfrak{q}) = \frac{\mathcal{S}ym(P^*)}{(\mathfrak{q})} \quad \text{and} \quad X(\mathfrak{q}) = \mathrm{Proj}\,(S(\mathfrak{q}))$$

- We say $(P, \mathfrak{q})$ is a **non degenerate**, if φ is an isomorphism.

Ch. 11. K-Theory of quadrics

- **Lemma:** Let R be a commutative ring, with $1/2 \in R$. Let $(P, \mathfrak{q})$ be a non degenerate quadratic R-modules. Then, $X(\mathfrak{q}) \longrightarrow \mathrm{Spec}\,(R)$ is smooth.
- To work with the sphere, we would have

$$\left\{ \begin{array}{ll|l} \mathfrak{q}_d = \sum_{i=0}^{d} X_i^2 & P = R^{d+1} & \dim X(\mathfrak{q}_d) = d-1 \\ \mathfrak{q}_d^s = \sum_{i=0}^{d} X_i^2 - T^2 & P = R^{d+2} & \dim X(\mathfrak{q}_d^s) = d \end{array} \right.$$

Ch. 11. K-Theory of quadrics

Theorem: Let R be a commutative ring, with $1/2 \in R$.
Let $(P, \mathfrak{q})$ be a non degenerate quadratic R-modules.
Assume $rank(P) = d + 1$.
We denote $\mathfrak{q}^s = \mathfrak{q} - T^2$ on $P \oplus R$. In fact

$$\begin{cases} (P \oplus R, \mathfrak{q}^s) = (P, \mathfrak{q}) \perp (R, -T^2), & \\ X(\mathfrak{q}) = (T = 0) \subseteq X(\mathfrak{q}^s) & \dim X(\mathfrak{q}^s) = d \\ U := (T \neq 0) \cong \mathrm{Spec}\,(A(\mathfrak{q})) & \dim A(\mathfrak{q}) = d \end{cases}$$

$$\text{where} \quad A(\mathfrak{q}) = \frac{\mathcal{S}ym(P^*)}{(\mathfrak{q} - 1)} \qquad \text{the sphere}$$

Ch. 11. K-Theory of quadrics

Assume R is regular. Then, we have long exact sequence

$$\longrightarrow K_n(X(\mathfrak{q})) \xrightarrow{\iota_*} K_n(X(\mathfrak{q}^s)) \longrightarrow K_n(A(\mathfrak{q})) \longrightarrow K_{n-1}(X(\mathfrak{q}))$$

$$\longrightarrow \cdots \longrightarrow \cdots \longrightarrow \cdots \longrightarrow$$

$$\longrightarrow K_0(X(\mathfrak{q})) \longrightarrow K_0(X(\mathfrak{q}^s)) \longrightarrow K_0(A(\mathfrak{q})) \longrightarrow 0$$

Ch. 11. K-Theory of quadrics

$$\text{Apply Swan's formula to}: \quad \begin{cases} X(\mathfrak{q}) \longrightarrow \operatorname{Spec}(R) \\ X(\mathfrak{q}^s) \longrightarrow \operatorname{Spec}(R) \end{cases}$$

We have the vertical identifications:

$$\begin{array}{ccc} \bigoplus_{n=0}^{d-2} K_q(R) \bigoplus K_q^{gr}(C(\mathfrak{q})) & & \bigoplus_{n=0}^{d-1} K_q(R) \bigoplus K_q^{gr}(C(\mathfrak{q}^s)) \\ \wr\downarrow & & \wr\downarrow \\ K_q(X(\mathfrak{q})) & \xrightarrow[\iota_*]{} & K_q(\mathbb{H}(X(\mathfrak{q}^s))) \end{array}$$

Ch. 11. K-Theory of quadrics

Here $C(\mathfrak{q})$ denotes the **Clifford algebra** of $(P, \mathfrak{q})$, which has a $\mathbb{Z}_2$-grading.

- Thus, we can write the above long exact sequence, in terms of K-groups of R and $C(\mathfrak{q})$, $C(q^s)$.
- In particular, we can compute the $K_0(A(\mathfrak{q}))$ of the affine spheres.
- With $R = \mathbb{R}$, it leads to the result that $K_0(A(\mathfrak{q}_d)) \cong KO(\mathbb{S}^d)$.

Ch. 11. K-Theory of quadrics

Further inspection, the above exact sequence reduces to:
Corollary: Assume R is regular. Then, there is an exact sequence,

$$\xrightarrow{\partial} K_n^{gr}(C(\mathfrak{q})) \xrightarrow{(\beta,-\varepsilon)_*} K_n(R) \oplus K_n(C(\mathfrak{q})) \xrightarrow{(\rho_1,\rho_2)_*} K_n(A(\mathfrak{q}))$$

where $K_n(C(\mathfrak{q})) = K_n(\mathscr{P}_r(C(\mathfrak{q})))$, the K-groups of the category of right projective $C(\mathfrak{q})$-modules (ungraded).

Ch. 11. K-Theory of quadrics

We reinterpret the functors:

$$\begin{cases} \beta : \mathscr{P}_{r,\mathbb{Z}_2}(C(\mathfrak{q})) \longrightarrow \mathscr{P}(R) & \beta(M) = M_1 \\ \varepsilon : \mathscr{P}_{r,\mathbb{Z}_2}(C(\mathfrak{q})) \longrightarrow \mathscr{P}_r(C(\mathfrak{q})) & \varepsilon(M) = M \text{ (ungraded)} \\ \rho_1 : \mathscr{P}(R) \longrightarrow \mathscr{P}(R(\mathfrak{q})) & \rho_1(M) = M \otimes A(\mathfrak{q}) \\ \rho_2 : \mathscr{P}_r(C(\mathfrak{q})) \longrightarrow \mathscr{P}(R(\mathfrak{q})) & \rho_2(N) = \Gamma(U, \mathfrak{U}_{d-1}(N)) \end{cases}$$