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Complete intersection K-theory and Chern classes

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Dedicated to my father

0. Introduction

The purpose of this paper is to investigate the theory of complete intersection in noetherian commutative rings from the K-Theory point of view. (By *complete intersection theory*, we mean questions like when/whether an ideal is the image of a projective module of appropriate rank.)

The paper has two parts. In part one (Section 1-5), we deal with the relationship between complete intersection and K-theory. The Part two (Section 6-8) is, essentially, devoted to construction projective modules with certain cycles as the total Chern class. Here Chern classes will take values in the Associated graded ring of the Grothedieck $\gamma - filtration$ and as well in the Chow group in the smooth case.

In this paper, all our rings are commutative and schemes are noetherian. To avoid unnecessary complications, we shall assume that all our schemes are connected.

For a noetherian scheme $X,\ K_0(X)$ will denote the Grothendieck group of locally free sheaves of finite rank over X. Whenever it make sense, for a coherent sheaf M over $X,\ [M]$ will denote the class of M in $K_0(X)$. We shall mostly be concerned with $X = SpecA$, where A is a noetherian commutative ring and in this case we shall also use the notation $K_0(A)$ for $K_0(X)$.

Discussion on Part One (Section 1-5)

For a noetherian commutative ring A of dimension n, we let

$$F_0K_0A = \{[A/I] \text{ in } K_0A :\ I \text{ is a locally complete intersection ideal of height } n\}.$$

In Sect. 1, we shall prove that F_0K_0A is a subgroup of K_0A. We shall call this subgroup F_0K_0A, the *zero cycle subgroup* of K_0A. We shall also see that (1.6), for a reduced affine algebra A over an algebraically closed field k, F_0K_0A is the subgroup generated by smooth maximal ideals of height n. The later subgroup was considered by Levine [Le] and Srinivas [Sr].

One of our main results (3.2) in Part One is that *for a noetherian commutative ring A of dimension n suppose that whenever I is a locally complete intersection ideal of height n with $[A/I] = 0$ in K_0A, there is a projective (respectively* stably free) A-module P *of* rank n *that maps onto I. Then for any locally complete intersection ideal I of* height n, *whenever $[A/I]$ is divisible by $(n-1)!$ in F_0K_0A, I is image of a projective A*-module Q *of* rank n *(respectively* with $(n-1)!([Q] - [A^n]) = -[A/I]$ in K_0A).

In [Mu2, (3.3)], Murthy proved that for a reduced affine algebra A over an algebraically closed field k, for an ideal I, if I/I^2 is generated by $n =$ dim A elements then I is image of a projective A-module of rank n.

In example (3.6), we show that for the coordinate ring

$$A = \mathbb{R}[X_0, X_1, X_2, X_3]/(X_0^2 + X_1^2 + X_2^2 + X_3^2 - 1)$$

of real 3-sphere, the ideal $I = (X_0 - 1, X_1, X_2, X_3)A$ is not the image of a projective A-module of rank 3, although $[A/I] = 0$ in K_0A.

Another interesting result (3.4) in this part is that *suppose that $f_1, f_2, \ldots, f_r$ is a regular sequence in a noetherian commutative ring A of dimension n and let Q be a projective A*-module *of rank r that maps onto $(f_1, \ldots, f_{r-1}, f_r^{(r-1)!})$. Then $[Q] = [Q_0 \oplus A]$ in K_0A for some projective A*-module Q_0 *of rank $r - 1$.*

When $dim\ A = n = r = rank\ Q$, this result(3.4) has interesting comparison with the corresponding theorem of Mohan Kumar [Mk1] for reduced affine algebras A over algebraically closed fields.

More generally we prove that (3.5) *suppose A is a noetherian commutative ring of* dimension n *and let J be a locally complete intersection ideal of* height $r \leqslant n$. *Assume that K_0A has no $(r-1)!$ torsion, $[A/J] = 0$ and J/J^2 has free generators of the form $f_1, f_2, \ldots, f_{r-1}, f_r^{(r-1)!}$ in J. Let Q be a projective A*-module *of* rank r *that maps onto J. Then $[Q] = [Q_0 \oplus A]$ in K_0A, for some projective A*-module Q_0 *of* rank $r - 1$.

For reduced affine algebras A of dimension n over algebraically closed fields k, and for $n = r$, (3.5) is a consequence of the theorem of Murthy [Mu2, Theorem 3.7]. Besides these results [Mk1,Mu2] (3.4) and (3.5) are the best in this context , even for affine algebras over algebraically closed fields. In fact, there is almost no result available in the case when rank is strictly less than the dimension of the ring.

In Sect. 1, we define and describe the zero cycle subgroup F_0K_0A of $K_0(A)$. In Sect. 2, for $k' = \mathbb{Z}$ or a field, we define the ring

$$A_n = A_n(k') = \frac{k'[S,T,U,V,X_1,\dots,X_n,Y_1,\dots,Y_n]}{(SU+TV-1, X_1Y_1+\cdots+X_nY_n-ST)}.$$

For our purposes, A_n serves like a "universal ring." Besides doing the construction of the "universal projective module" (2.6), we compute the K_0A_n, the Chow Group of A_n and we comment on the higher K-groups of A_n.

All the results in Section 3 discussed above follows from a key Theorem (3.1). In Section 4, we give the proof of (3.1). In Section 5, we give some more applications of (3.1).

Discussion on Part Two (Section 6-8)

The purpose of this part of the paper is to construct projective modules of appropriate rank that have certain cycles as its Chern classes and to consider related questions.

For a noetherian scheme X of dimension n, $\Gamma(X) = \bigoplus_{i=1}^n \Gamma^i(X)$ will denote the graded ring associated to the Grothendieck γ-filtration of the Grothendieck group $K_0(X)$ and $CH(X) = \bigoplus_{i=1}^n CH^i(X)$ will denote the Chow group of cycles of X modulo rational equivalence.

Our main construction (8.3) is as follows: *suppose $X = SpecA$ is a Cohen-Macaulay scheme of dimension n and $r \geqslant r_0$ are integers with* $2r_0 \geqslant n$ and $n \geqslant r$. *Given a projective A*-module Q_0 *of rank $r_0 - 1$ and a sequence of locally complete intersection ideals I_k of height k for $k = r_0$ to r such that*

(1) *the restriction $Q_0|Y$ is trivial for all locally complete intersection subschemes Y of codimension at least r_0 and*

(2) *for $k = r_0$ to r I_k/I_k^2 has a free set of generators of the type* $f_1, \dots, f_{k-1}, f_k^{(k-1)!}$ in I_k,

then there is a projective A-module Q_r *of rank r such that*

(1) *for $1 \leqslant k \leqslant r_0 - 1$ the kth Chern class of Q_0 and Q_r are same and*

(2) *for k between r_0 and r the kth Chern class of Q_r is given by the cycle of A/I_k, upto a sign.*(Here Chern classes take values in $\Gamma(X) \bigotimes \mathbb{Q}$ or in the Chow group, if X is nonsingular over a field).*More precisely, we have*

$$[Q_r] - r = [Q_0] - (r_0 - 1) + \sum_{k=r_0}^{r} [A/J_k]$$

in $K_0(X)$ where J_k is a locally complete intersection ideal of height k with $[A/I_k] = -(k-1)![A/J_k]$.

Inductive arguments are used to do the construction of Q_r in theorem (8.3). Conversely, we prove theorem (8.2):

*let A be a commutative noetherian ring of dimension n and $X = SpecA$.Let J be a locally complete intersection ideal of height r so that J/J^2 has a free set of generators of the form $f_1, f_2, \ldots, f_{r-1}, f_r^{(r-1)!}$. Let Q be projective A-*module *of rank r that maps onto J.Then there is a projective A* -module Q_0 *of rank $r-1$ such that the first $r-1$ Chern classes of Q and Q_0 are same.*(Here again Chern classes take values in $\Gamma(X)$ or in the Chow group,if X is nonsingular over a field). *Infact, if $K_0(X)$ is torsion free then $[Q_0]$ is unique in $K_0(X)$.*

Both in the statements of theorem (8.2) and (8.3), we considered locally complete intersection ideals J of height r so that J/J^2 has free set of generators of the form $f_1, f_2, \ldots, f_{r-1}, f_r^{(r-1)!}$ in J. For such an ideal J, $[A/J] = (r-1)![A/J_0]$, for some locally complete intersection ideal J_0. Consideration of such ideals are supported in theorem (8.1):

Let A be a noetherian commutative ring of dimension n and $X = SpecA$. Assume that $K_0(X)$ has no $(n-1)!$-torsion. *Let I be locally complete intersection ideal of height n that is image of a projective A*-module *Q of rank n.Also suppose that Q_0 is an A*-module *of rank $n-1$ so that the first $n-1$ Chern classes of Q and Q_0 are same.Then $[A/I]$ is divisible by $(n-1)!$.*

For a variety X,what cycles of X,in $\Gamma(X)$ or in the Chow group,that may appear as the total Chern class of a locally free sheaf of appropriate rank had always been an interesting question, although not much is known in this direction.

For affine smooth three folds $X = SpecA$ over algebraically closed fields, Mohan Kumar and Murthy[MM] proved that (see 8.9) if c_k is a cycle in $CH^k(X)$, for $k = 1, 2, 3$ then there are projective A-modules Q_k of rank k so that
(1) $total\ Chern\ class\ of\ Q_1\ is\ 1 + c_1$
(2) $the\ total\ Chern\ class\ of\ Q_2\ is\ 1 + c_1 + c_2,$
(3) $the\ total\ Chern\ class\ of\ Q_3\ is\ 1 + c_1 + c_2 + c_3.$

We give a stronger version (8.10) of this theorem (8.9) of Mohan Kumar and Murthy [MM]. Our theorem (8.10) applies to any smooth three fold X over any field such that $CH^3(X)$ is divisible by 2.

Murthy[Mu2] also proved that if $X = SpecA$ is a smooth affine variety of dimension n over an algebraically closed filed k and c_n is a codimension n cycle in the Chow group of X then there is a projective A-module Q of rank n so that the total Chern class of Q is $1 + c_n$. We give a stronger version (8.7) of this theorem of Murthy [Mu2]. This version (8.7) of the theorem applies to all smooth affine varieties X of dimension n, over any field, so that $CH^n(X)$ is divisible by $(n-1)!$.

Murthy [Mu2] also proved : suppose that $X = SpecA$ is a smooth affine variety of dimension n over an algebraically closed field k. For $i = 1$ to n let c_i be a codimension i cycle in the Chow group of X. Then there is a projective A-module Q_0 of rank $n-1$ with total Chern class $1 + c_1 + \cdots + c_{n-1}$ if and only if there is a projective A-module Q of rank n with total Chern class $1 + c_1 + \cdots + c_n$. We also give an alternative proof (8.8) of this theorem of Murthy [Mu2] .

Besides these results [MM,Mu2] not much else is known in this direction. Our results in Sect. 8 apply to any smooth affine variety over any field and also consider codimesion r cycles where r is strictly less than the dimension of the variety. Consideration of Chern classes in the Associated graded ring of the Grothendieck γ-filtration in the nonsmooth case is, possibly, the only natural thing to do because the theory of Chern classes in the Chow group is not available in such generality. Such consideration of Chern classes in the Associated graded ring of the Grothendieck γ-filtration was never done before in this area .

In Sect. 6, we set up the notations and other formalism about the Grothendieck Gamma filtration, Chow groups and Chern classes. In this section we also give an example of a smooth affine variety X for which the Grothedieck Gamma filtration of $K_0(X)$ and the filtration by the codimension of the support do not agree.

In Sect. 7, we set up some more preliminaries. Our main results of the Part Two of the paper are in Sect. 8.

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Part one : Section 1-5

Complete intersection and K-theory

In this part, we investigate the relationship between complete intersection and K-theory.

1. The Zero Cycle Subgroup

For a noetherian commutative ring A, $K_0(A)$ will denote the Grothendieck group of projective A-modules of finite rank. We define $F_0K_0A = \{[A/I]$ in $K_0A : I$ is a locally complete intersection ideal of height n = dim$A\}$.

In this section, we shall prove that F_0K_0A is a subgroup of K_0A. We call this subgroup F_0K_0A, the *zero cycle subgroup*. We shall also prove that if A is an affine algebra over an algebraically closed field k, this notation F_0K_0A is consistent with the notation used by Levine [Le] and Srinivas [Sr] for the subgroup of K_0A generated by $[A/\mathfrak{M}]$, where $\mathfrak{M}$ is a smooth maximal ideal in A.

Theorem 1.1. *Suppose A is a noetherian commutative ring of dimension n. Then F_0K_0A is a subgroup of K_0A.*

The proof of (1.1) will follow from the following Lemmas.

Lemma 1.2. $F_0K_0A = -F_0K_0A$.

Proof. Suppose I is a locally complete intersection ideal of height n = dim A and $x = [A/I]$ is in F_0K_0A.

Let $I = (f_1, f_2, \dots, f_n) + I^2$. By induction, we shall find $f_1', f_2', \dots, f_r'$ in I such that

(1) $(f_r', \dots, f_r', f_{r+1}, \dots, f_n) + I^2 = I$,
(2) $(f_1', \dots, f_r')$ is a regular sequence.

Suppose we have picked $f_1', f_2', \dots, f_r'$ as above and $r < n$. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the associated primes of $(f_1', f_2', \dots, f_r')$. If I is contained in $\mathfrak{p}_1$, then height $\mathfrak{p}_1 = n$ and since $I_{\mathfrak{p}_1}$ is complete intersection of height n, $A_{\mathfrak{p}_1}$ is Cohen-Macaulay ring of height n. This contradicts that $\mathfrak{p}_1$ is associated prime of $(f_1', \dots, f_r')$. So, I is not contained in $\mathfrak{p}_i$ for $i = 1$ to s. Let $\{P_1, \dots, P_t\}$ be maximal among $\{\mathfrak{p}_1, \dots, \mathfrak{p}_s\}$ and assume that f_{r+1} is in $P_1, \dots, P_{t_0}$ and not in $P_{t_0+1}, \dots, P_t$. Let a be in $I^2 \cap P_{t_0+1} \cap \dots \cap P_t \setminus P_1 \cup P_2 \cup \dots \cup P_{t_0}$. Let $f_{r+1}' = f_{r+1} + a$. Then f_{r+1}' does not belong to P_i for $i = 1$ to t and hence also does not belong to $\mathfrak{p}_1, \dots, \mathfrak{p}_s$. Hence we have that

(1) $(f_1', \dots, f_r', f_{r+1}', f_{r+2}, \dots, f_n) + I^2 = I$ and
(2) $f_1', \dots, f_r', f_{r+1}'$ is a regular sequence.

Therefore, we can find a regular sequence $f_1', f_2', \dots, f_n'$ such that $I = (f_1', \dots, f_n') + I^2$. So, $(f_1', f_2', \dots, f_n') = I \cap J$ for some ideal J with $I + J = A$. Since $f_1', \dots, f_n'$ is a regular sequence, J is locally complete intersection ideal and $[A/I] + [A/J] = [A/(f_1', f_2', \dots, f_n')] = 0$. Hence $[A/J] = -x$ is in F_0K_0A. So, the proof of (1.2) is complete.

Lemma 1.3. *Suppose A is a noetherian commutative ring of height n and I is a locally complete intersection ideal of height n. Let $\mathfrak{M}_1, \dots, \mathfrak{M}_k$ be* *maximal ideals that does not contain I. There are $f_1, f_2, \ldots, f_n$ such that*

(1) $f_1, \ldots, f_n$ is a regular sequence,

(2) $I = (f_1, \ldots, f_n) + I^2$,

(3) for a maximal ideal $\mathfrak{M}$, if $(f_1, \ldots, f_n)$ is contained in $\mathfrak{M}$, then $\mathfrak{M} \neq \mathfrak{M}_i$ for $i = 1$ to k.

Proof. As in the proof of (1.2), we can find a regular sequence $f_1, \ldots, f_n$ such that $I = (f_1, \ldots, f_n) + I^2$. We readjust f_n to avoid $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ as follows. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_s$ be the associated primes of $(f_1, \ldots, f_{n-1})$. Then I is not contained in $\mathfrak{p}_i$ for $i = 1$ to s. Let $\{P_1, P_2, \ldots, P_t\}$ be maximal among $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_s, \mathfrak{M}_1, \ldots, \mathfrak{M}_k\}$. Assume that f_n is in $P_1, \ldots, P_{t_0}$ and not in $P_{t_0+1}, \ldots, P_t$. Let a be in $I^2 \cap P_{t_0+1} \cap \cdots \cap P_t \setminus P_1 \cup P_2 \cup \cdots \cup P_{t_0}$ and $f_n' = f_n + a$. Then f_n' is not in $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$. So,

(1) $f_1, f_2, \ldots, f_{n-1}, f_n'$ is a regular sequence

(2) $I = (f_1, f_2, \ldots, f_{n-1}, f_n') + I^2$ and

(3) if $(f_1, \ldots, f_{n-1}, f_n') \subseteq \mathfrak{M}$ for a maximal ideal $\mathfrak{M}$ then $\mathfrak{M} \neq \mathfrak{M}_i$ for $i = 1$ to k.

This completes the proof of (1.3)

Lemma 1.4. *Let A be as in* (1.1). *Then F_0K_0A is closed under addition.*

Proof. Let x and y be in F_0K_0A. Then $x = [A/I]$ and by (1.2), $y = -[A/J]$, where I and J are locally complete intersection ideals of height n. Let $\{\mathfrak{M}_1, \ldots, \mathfrak{M}_k\} = V(I) \setminus V(J)$, the maximal ideals that contain I and do not contain J. By (1.3), there is a regular sequence $f_1, f_2, \ldots, f_n$ such that $J = (f_1, \ldots, f_n) + J^2$ and for maximal ideals $\mathfrak{M}$ that contains $(f_1, \ldots, f_n), \mathfrak{M} \neq \mathfrak{M}_i$ for $i = 1$ to k.

Let $(f_1, \ldots, f_n) = J \cap J'$, where $J + J' = A$ and J' is a locally complete intersection ideal of height n. Then $y = -[A/J] = [A/J']$. Also note that $I + J' = A$. Hence $x + y = [A/I] + [A/J'] = [A/IJ']$, and IJ' is a locally complete intersection ideal of height n. So the proof of (1.4) is complete.

Clearly, the proof of Theorem (1.1) is complete by (1.2) and (1.4). Now we proceed to prove that for reduced affine algebras A over a field k, F_0K_0A is generated by regular points.

Theorem 1.5. *Suppose A is a reduced affine algebra over a field k of dimension n. Then F_0K_0A is generated by the classes $[A/\mathfrak{M}]$, where $\mathfrak{M}$ runs through all the regular maximal ideals of height n.*

Proof. Since the regular locus of A is open (see [K]), there is an ideal J of A such that $V(J)$ is the set of all prime ideals P such that A_P is not regular. Since A is reduced, height $J \geqslant 1$.

Let G be the subgroup of K_0A, generated by all classes $[A/\mathfrak{M}]$, where $\mathfrak{M}$ is a regular maximal ideal of A of height n. Clearly G is contained in F_0K_0A. Now let $x = [A/I]$ be in F_0K_0A, with I a locally complete intersection ideal of height n. Let $I = (f_1, f_2, \dots, f_n) + I^2$. By induction we shall find $f'_1, f'_2, \dots, f'_r$ for $r \leqslant n$, such that

(1) $I = (f'_1, \dots, f'_r, f_{r+1}, \dots, f_n) + I^2$,

(2) $(f',_1, f'_2, \dots, f'_r)$ is a regular sequence and

(3) for a prime ideal P of A, if $J + (f'_1, \dots, f'_r)$ is contained in P then either height $P > r$ or I is contained in P.

We only need to show the inductive step. Suppose we have picked $f'_1, \dots,$ f'_r as above. Let $\mathfrak{p}_1, \dots \mathfrak{p}_k$ be the associated primes of $(f'_1, \dots f'_r)$ and let $Q_1, Q_2, \dots, Q_s$ be the minimal primes over $(f'_1, f'_2, \dots, f'_r) + J$ so that I is not contained in Q_i for $i = 1$ to s. So, we have height $Q_i > r$. As before, we see that I is not contained in $\mathfrak{p}_i$ for $i = 1$ to k.

Let $\{P_1, P_2, \dots, P_t\}$ be the maximal elements in $\{\mathfrak{p}_1, \dots, \mathfrak{p}_k, Q_1, \dots,$ $Q_s\}$ and let f_{r+1} be in $P_1, \dots, P_{t_0}$ and not in $P_{t_0+1}, \dots, P_t$. Let a be in $I^2 \cap P_{t_0+1} \cap \dots \cap P_t \setminus P_1 \cup P_2 \cup \dots \cup P_{t_0}$. Write $f'_{r+1} = f_{r+1} + a$. Then f'_{r+1} will satisfy the requirement.

Hence we have a sequence $f'_1, f'_2, \dots, f'_n$ such that

(1) $I = (f'_1, \dots, f'_n) + I^2$,

(2) $f'_1, f'_2, \dots, f'_n$ is a regular sequence and

(3) if a maximal ideal $\mathfrak{M}$ contains $(f'_1, \dots f'_n) + J$ then I is contained in $\mathfrak{M}$. If $(f'_1, \dots, f'_n) = I \cap I'$, then $I + I' = A$ and I' is a locally complete intersection ideal of height n. Also, if a maximal ideal $\mathfrak{M}$ contains I', then $\mathfrak{M}$ is a regular maximal ideal of height n. So, $[A/I']$ is in G and hence $x = [A/I] = -[A/I']$ is also in G. The proof of (1.5) is complete.

Remark 1.6. From the proof of (1.5), it follows that (1.5) is valid for any noetherian commutative ring A such that the singular locus of spec A is contained in a closed set $V(J)$ of codimension at least one. Similar arguments work for smooth ideals.

2. The Universal Constructions

For $k' = \mathbb{Z}$ or a field, we let

$$K = K(k') = \frac{k'[S, T, U, V]}{(SU + TV - 1)}$$

$$A_n = A_n(k') = \frac{k'[S, T, U, V, X_1, \dots, X_n, Y_1, \dots, Y_n]}{(SU + TV - 1, X_1Y_1 + \dots + X_nY_n - ST)}$$

$$B_n = B_n(k') = \frac{k'[T, X_1, \dots, X_n, Y_1, \dots, Y_n]}{(X_1Y_1 + X_2Y_2 + \dots + X_nY_n - T(1 + T))}$$

By the natural map $A_n \to B_n$, we mean the map that sends $T \to T, S \to 1+T, U \to 1, V \to -1$. (We will continue to denote the images of upper case letter variables in A_n or B_n, by the same symbol).

The ring B_n, was considered by Jouanlou [J]. Later B_n was further used by Mohan Kumar and Nori [Mk2] and Murthy [Mu2]. The purpose of this section is to establish that A_n behaves much like B_n.

The Grothendieck Group and the Chow Group of A_n

For a ring A and X = Spec $A, K_0(A)$ or $K_0(X)$ (*respectively* $G_0(A)$ or $G_0(X)$) will denote the Grothendieck Group of finitely generated projective modules (*respectively* finitely generated modules) over A. $CH^k(A)$ or $CH^k(X)$ will denote the Chow Group of cycles of codimension k modulo rational equivalence and $CH(X) = \bigoplus CH^k(X)$ will denote the total Chow group of X.

Proposition 2.1. *Let $\lambda_n = [A_n/(X_1, X_2, \dots, X_n, T)]$ in $G_0(A_n)$. Then $G_0(A_n)$ is freely generated by $\epsilon_n = [A_n]$ and λ_n. In fact, the natural map $G_0(A_n) \to G_0(B_n)$ is an isomorphism.*

Proof. We proceed by induction on n. If $n = 0$, then $A_0 \approx A_0/(S) \times A_0/(T)$. Since $A_0/(S) \approx A_0/(T) \approx k'[S^{\pm 1}, V]$, the proposition holds in this case.

Now assume $n > 0$. We have $A_{n_{X_n}} \approx K[X_1, \dots, X_{n-1}, X_n^{\pm 1}, Y_1, \dots, Y_{n-1}]$ and $A_n/(X_n) \approx A_{n-1}[Y_n]$. Since $G_0(K) \approx \mathbb{Z}$ (see [Sw1], Sect. 10), $G_0(A_{n_{X_n}}) \approx \mathbb{Z}$ and also by induction $G_0(A_n/(X_n)) \approx G_0(A_{n-1}[Y_n]) \approx G_0(A_{n-1})$ is generated by $[A_{n-1}]$ and λ_{n-1}. Now we have the exact sequence $G_0(A_n/(X_n)) \xrightarrow{i_*} G_0(A_n) \xrightarrow{j^*} G_0(A_{n_{X_n}}) \to 0$. Since the $i_*(\lambda_{n-1}) = \lambda_n$ and $i_*([A_{n-1}]) = 0, G_0(A_n)$ is generated by λ_n and $[A_n]$.

It is also easy to see that the natural map $G_0(A_n) \to G_0(B_n)$ sends λ_n to $\beta_n = [B_n/(X_1, \dots, X_n, T)]$ and $[A_n]$ to $[B_n]$. Since β_n and $[A_n]$ are free generators of $G_0(B_n)$ (see [Sw1, §10]/[Mu2]), λ_n and $[A_n]$ are free generators of $G_0(A_n)$. This completes the proof of (2.1).

Proposition 2.2. *Let λ'_n be the cycle defined by $A_n/(X_1, \dots X_n, T)$ in $CH(A_n)$ and let $\epsilon'_n = [Spec\ A_n]$ be the cycle of codimension zero. Then $CH(A_n)$ is freely generated by ϵ'_n and λ'_n. That means $CH^j(A_n) = 0$ for $j \neq 0, n, CH^0(A_n) = \mathbb{Z}\epsilon'_n \approx \mathbb{Z}$ and $CH^n(A_n) = \mathbb{Z}\epsilon'_n \approx \mathbb{Z}$.*

Before we prove (2.2), we prove the following easy lemma.

Lemma 2.3. $CH(K) = \mathbb{Z}[Spec\ K]$.

Proof. Note that $K/UK \approx k'[T^{\pm 1}, S]$ and $K_U \approx k'[T, U^{\pm 1}, V]$. Now the lemma follows from the exact sequence $CH_j(K/UK) \to CH_j(K) \to$

$CH_j(K_U) \to 0$ for all j. (Here we use the notation $CH_j(S) = CH^{dim\ x-j}$ $(X))$.

Proof of (2.2). For $n = 0, A_0 = A_n \approx K/(ST) \approx K/(S) \times K/(T) \approx k'[T^{\pm 1}, U] \times k'[S^{\pm 1}, V]$. $CH^0(A_0) \approx \mathbb{Z}[V(S)] \oplus \mathbb{Z}[V(T)] \approx \mathbb{Z}\epsilon'_0 \oplus \mathbb{Z}\lambda'_0$ and $CH^j(A_0) = 0$ for all $i > 0$. So, the Proposition 2.2 holds for $n = 0$. Let

$$d_n = dim\ A_n = \begin{cases} 2n+2 \text{ if } k' \text{is a field} \\ 2n+3 \text{ if } k' = \mathbb{Z}. \end{cases}$$

Let $n > 0$ and assume that the proposition holds for $n-1$. Since $A_{n_{X_n}} \approx K[X_1, \ldots, X_{n-1}, X_n^{\pm 1}, Y_1, \ldots, Y_{n-1}]$ and $A_n/(X_n) \approx A_{n-1}[Y_n]$, $CH(A_{n_{X_n}}) \approx \mathbb{Z}[Spec\ A_{n_{X_n}}]$ and $CH(A_n/(X_n) \approx CH(A_{n-1}[Y_n]) \approx CH(A_{n-1})$. By induction it follows that $CH^j(A_n/(X_n) = 0$ for $j \neq 0, n-1$ and $CH^{n-1}(A_n/(X_n)$ is freely generated by $[A_n/(X_1, \ldots, X_n, T)]$.

Now consider the exact sequence $CH^{j-1}(A_n/(X_n)) \to CH^j(A_n) \to CH^j(A_{n_{X_n}}) \to 0$. It follows that for $j \neq 0, n, CH^j(A_n) = 0$ and clearly, $CH^0(A_n)$ is freely generated by $\epsilon'_n = [Spec\ A_n]$. Also $CH^n(A_n)$ is generated by the image of $[A_n/(X_1, \ldots, X_n, T)]$, which is λ'_n. Since the natural map $CH^n(A_n) \to G_0(A_n)$ maps λ'_n to λ_n and λ_n is a free generator, it follows that λ'_n is also torsion free. Hence $CH^n(A_n) = \mathbb{Z}\lambda'_n \approx \mathbb{Z}$. This completes the proof of (2.2).

Higher K-Groups of A_n

Much of this section is inspired by the arguments of Murthy [Mu3] and Swan [Sw1]. Again for a ring A, $G_i(A)$ will denote the i-th K-group of the category of finitely generated A-modules. For a subring K of A, $\widetilde{G}_i(A, K)$ will denote the cokernel of the map $G_i(K) \to G_i(A)$. As also explained in [Sw1], if there is an augmentation $A \to K$ with finite tordimension, then $0 \to G_i(K) \to G_i(A) \to \widetilde{G}_i(A, K) \to 0$ is a split exact sequence.

Following is a remark about the higher K-groups of A_n.

Theorem 2.4. *Let k be a field or $\mathbb{Z}$ and $K = K(k)$ and $A_n = A_n(k)$. Then*

(1) $G_i(K) \approx G_i(k) \oplus G_{i-1}(k)$ for all $i \geqslant 0$

(2) for $i \geqslant 0, n \geqslant 1, \widetilde{G}_i(A_{n,k}) \approx \widetilde{G}_i(A_1, K)$ and $0 \to G_i(K) \to G_i(A_n) \to \widetilde{G}_i(A_n, K) \to 0$ is split exact,

(3) There is a long exact sequence $\cdots \to G_i(k[T^{\pm 1}]) \oplus G_i(k[S^{\pm 1}]) \to G_i(A_1) \to G_i(K[X^{\pm 1}]) \xrightarrow{\partial} G_{i-1}(k[T^{\pm 1}]) \oplus G_{i-1}(k[S^{\pm 1}]) \to G_{i-1}(A_1)) \to \ldots$

Proof. The statement (1) is a theorem of Jouanolou [J]. To prove (2), note that all rings we consider are regular and that $K \to A_n$ has an augmentation for $n \geqslant 1$. Also note that for $n \geqslant 2, K[X_n] \to A_n$ is a flat extension. So, it induces a map of the localization sequences

(2.5)

$$\begin{array}{ccccccccc}
 & & & & G_i(K) & & & & \\
 & & & & ss & & & & \\
\cdots \to & G_i(K) & \to & G_i(K[X_n]) & \to G_i(K[X_n^{\pm 1}]) & \to & G_{i-1}(K) & \to & \dots \\
 & \downarrow & & \downarrow & \downarrow ss & & \downarrow & & \\
\cdots \to & G_i(A_n/X_n) & \to & G_i(A_n) & \to G_i(A_{n_{X_n}}) & \to & G_i(A_n/X_n) & \to & \dots \\
 & ss & & & & & ss & & \\
 & G_i(A_{n-1}) & & & & & G_{i-1}(A_{n-1}) & &
\end{array}$$

Also note that $0 \to G_i(K) \to G_i(K[X^{\pm 1}]) \to G_{i-1}(K) \to 0$ is a split exact sequence ([Q]). This will induce an exact sequence $\to G_i(K) \to G_i(A_{n-1}) \to \widetilde{G}_i(A_n, K) \to G_{i-1}(K) \to \dots$. Since $G_i(K) \to G_i(A_{n-1})$ is splits, it follows that $0 \to G_i(K) \to G_i(A_{n-1}) \to \widetilde{G}_i(A_n, K) \to 0$ is split exact. Hence $\widetilde{G}_i(A_{n-1}, K) \approx \widetilde{G}_i(A_n, K)$. This establishes statement (2). The statement (3) is the is immediate consequence of the localization sequence ([Q]) $\to G_i(A_1/(X_1) \to G_i(A_1) \to G_i(A_{1_{X_1}}) \to G_{i-1}(A_1/(X_1)) \to \dots$. This completes the proof of (2.4).

Construction of the Universal Projective Module

Under this subheading we construct a projective module over A_n, which will be useful in the later sections. This construction is similar to the construction of Mohan Kumar and M. V. Nori [Mk2] over B_n.

Proposition 2.6. *Let J_n be the ideal $(X_1, X_2, \dots, X_n, T)$ in A_n. Then*

(1) for $n \geqslant 3$, there is no projective A_n-module of rank n that maps onto J_n.

(2) There is a projective A_n-module P of rank n that maps onto the ideal $J'_n = (X_1, \dots, X_{n-1})A_n + J_n^{(n-1)!}$ such that $([P]) - n) = [A/J_n] = -\lambda_n$ in $G_0(A_n)$.

Proof. The proof of the statement (1) is similar to the argument in [Mk2] or this can also be seen by tensoring with B_n and using the result in [Mk2].

To prove statement (2), note that $J_{n_S} = (X_1, \dots, X_n)$ and $J'_{n_S} = (X_1, \dots, X_{n-1}, X_n^{(n-1)!})$. Also, since $(X_1, \dots, X_{n-1}, X_n^{(n-1)!})$ is an unimodular row in $A_{n_{ST}}$, by Suslin's Theorem ([S]), there is an $n \times n$-matrix γ in $\mathbb{M}_n(A_n)$ such that $det(\gamma) = (ST)^u$ for some $u \geqslant 0$ and the first column of γ is the transpose of $(X_1, X_2, \dots, X_{n-1}, X_n^{(n-1)!})$. Let $f_1 : A^n_{n_S} \to J'_{n_S}$ be the map that sends the standard basis $e_1, \dots, e_n$ of $A^n_{n_S}$ to $X_1, X_2, \dots, X_{n-1}, X_n^{(n-1)!}$ and let $f_2 : A^n_{n_T} \to J_{n_T} \approx A_{n_T}$ be the map that sends the standard basis $e_1, \ldots, e_n$ to $1, 0, 0, \ldots, 0$. As in the paper of Boratynski [B], by patching f_1 and f_2 by γ, we get a surjective map $P \to J'_n$, where P is a projective A_n-module of rank n.

Now we wish to establish that $([P] - [A_n^n]) = -\lambda_n$. By tensoring with $\mathbb{Q}$, in case $k = \mathbb{Z}$, we can assume that k is a field.

The rest of the argument is as in Murthy's paper [Mu2]. Let $[P] - [A_n^n] = m\lambda_n$. So, $C_n([P] - [A_n^n]) = (-1)^n[V(J'_n)] = (-1)^n(n-1)!\lambda'_n$. Also, by the Riemann-Roch theorem, $C_n([P] - [A_n^n]) = mC_n(\lambda_n) = m(-1)^{n-1}(n-1)!\lambda'_n$. Hence it follows from (2.2) that $m = -1$. Hence $P - [A_n^n] = -\lambda_n$.

3. The main results in part one

Our main results follow from the following central theorem.

Theorem 3.1. *Let A be a commutative noetherian ring of dimension n and let I and J_0 be two ideals that contain nonzero divisors and $I + J_0 = A$. Assume that J_0 is a locally complete intersection ideal of height r with $J_0 = (f_1, \ldots, f_r) + J_0^2$ and let $J = (f_1, \ldots, f_{r-1}) + J_0^{(r-1)!}$. Suppose Q is a projective A-module of rank r and $\varphi : Q \to IJ$ is a surjective map. Then*

(i) there is a projective A-module P of rank r that maps onto J with $[P] - [A^r] = -[A/J_0]$ in $K_0(A)$;

(ii) further, there is a surjective map from $Q \oplus A^r$ onto $I \oplus P$;

(iii) in particular, there is a projective A-module Q' of rank r that maps onto I and $[Q'] = [Q] + [A/J_0]$ in $K_0(A)$.

In the rest of this section we shall use this theorem (3.1) to derive its main consequences and the proof of (3.1) will be given in the next section.

Theorem 3.2. *Let A be a noetherian commutative ring of dimension $n \geqslant 1$. Also assume that for locally complete intersection ideals I of height n, whenever $[A/I] = 0$ in $K_0(A)$, I is an image of a projective (*respectively, *with stably free) A-module Q of rank n.*

*Then for locally complete intersection ideals I of height n, if $[A/I]$ is divisible by $(n-1)!$ in F_0K_0A then I is image of a projective A-module Q' (*respectively, *with $(n-1)!([Q'] - n) = -[A/I]$) of rank n.*

Remark 3.3. If A is a reduced affine algebra over an algebraically closed field, Murthy [Mu2] proved that for any ideal I of A, if I/I^2 is generated by $n = \dim A$ elements then I is an image of a projective A-module of rank n.

Proof of (3.2). Let I be a locally complete intersection ideal of height n, so that $[A/I]$ is divisible by $(n-1)!$ in F_0K_0A. Let $[A/I] = (n-1)![A/J]$ in F_0K_0A.

Let $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ be maximal ideals that contains I and that does not contain J. By Lemma (1.3), we can find a locally complete intersection ideal J' of height n such that $[A/J] = -[A/J']$ and $I + J' = A$. Now let $J' = (f_1, ..., f_n) + J'^2$ and $J'' = (f_1, \ldots, f_{n-1}) + J'^{(n-1)!}$. So, $[A/I] = (n-1)![A/J] = -(n-1)![A/J''] = -[A/J'']$ and $I + J'' = A$. Hence $[A/IJ''] = 0$. By hypothesis, there is a projective (*respectively*, stably free) A-module Q of rank n that maps onto IJ''. By (3.1) there is a projective module Q' of rank r that maps onto I and $[Q'] = [Q] + [A/J']$. Hence also $(n-1)!([Q'] - [Q]) = (n-1)![A/J'] = -[A/I]$. So,the proof of (3.2) is complete.

Our next two applications (3.4, 3.5) of (3.1) are about splitting projective modules.

Theorem 3.4. *Let A be noetherian commutative ring and let $f_1, f_2, \ldots, f_r$ be a regular sequence. Let Q be projective $A - module$ of rank r that maps onto $(f_1, \ldots, f_{r-1}, f_r^{(r-1)!})$. Then $[Q] = [Q_0 \oplus A]$ for some projective $A - module$ Q_0 of rank $r-1$.*

The proof of (3.4) is immediate from (3.1) by taking $J_0 = (f_1, f_2, \ldots, f_r)$ and $I = A$.

Following is a more general version of (3.4).

Theorem 3.5. *Let A be a noetherian commutative ring of dimension n and let J be a locally complete intersection ideal of height $r \geqslant 1$, such that J/J^2 has free generators of the type $f_1, f_2, \ldots, f_{r-1}, f_r^{(r-1)!}$ in J. Suppose $[A/J] = 0$ in $K_0(A)$ and assume $K_0(A)$ has no $(r-1)!$ torsion. Then, for a projective A-module Q of rank r, if Q maps onto J, then $[Q] = [Q_0] + 1$ in $K_0(A)$ for some projective A-module Q_0 of rank $r-1$.*

Proof. First note that we can assume that $f_1, f_2, \ldots, f_r$ is a regular sequence. We can find an element s in J such that $s(1+s) = f_1g_1 + f_2g_2 + \cdots + f_{r-1}g_{r-1} + f_r^{(r-1)!}g_r$ for some $g_1, g_2, \ldots g_r$. Let $J_0 = (f_1, f_2, \ldots, f_{r-1}, f_r, s)$. Then J_0 is a locally complete intersection ideal of height r and $J = (f_1, \ldots, f_{r-1}) + J_0^{(r-1)!}$. Let $I = A$. Then by (3.1), there is a projective A-module Q' of rank r that maps onto A and $[Q'] = [Q] + [A/J_0] = [Q]$. Since $Q' = Q_0 \oplus A$ for some Q_0, the theorem (3.5) is established.

Remark 3.6. For reduced affine algebras A over algebraically closed fields k, Murthy [Mu2] proved a similar theorem for $r = n = \dim A \geqslant 2$. In that case, if $\operatorname{char} k = 0$ or $\operatorname{char} k = p \geqslant n$ or A is regular in codimension 1, then F_0K_0A has no $(n-1)!$-torsion. (See [Le], [Sr], [Mu2])

Before we close this section we give some examples.

Example 3.7. Let $A = \mathbb{R}[X_0, X_1, X_2, X_3]/(X_0^2 + X_1^2 + X_2^2 + X_3^2 - 1)$ be the coordinate ring of the real 3-sphere S^3. Then $K_0(A) = \mathbb{Z}$ (see [Hu] and [Sw2]) and $CH^3(A) = \mathbb{Z}/2\mathbb{Z}$ generated by a point [CF]. Since, in this case for any projective A-module Q of rank 3, the top Chern Class $C_3(Q) = 0$ in $CH^3(A)$, no projective A-module will map onto the ideal $I = (X_0 - 1, X_1, X_2, X_3)A$. This is a situation, when $[A/I] = 0$ in $K_0(A)$, but I is not an image of a projective module of rank 3.

4. Proof of theorem 3.1

In this section we give the proof of Theorem 3.1. First we state the following easy lemma.

Lemma 4.1. *Suppose A is a noetherian commutative ring and I and J are two ideals that contain nonzero divisors. Let $I + J = A$. Then we can find a nonzero divisor s in I such that $(s, J) = a$.*

Now we are ready to prove (3.1).

Proof of (3.1). The first part of the proof is to find a nonzero divisor s in I such that

(1) $(s, J_0) = A$,

(2) after possibly modifying $f_1, \ldots, f_r$, we have $sJ_0 \subseteq (f_1, \ldots, f_r)$ and

(3) Q_s is free with basis $e_1, \ldots, e_r$ such that $\varphi_s(e_1) = f_1, \ldots, \varphi_s(e_{r-1}) = f_{r-1}$ and $\varphi_s(e_r) = f_r^{(r-1)!}$.

First note that there is a nonzero divisor s_1 in I such that $As_1 + J = A$. Now let $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_r$ be the associated primes of A_{s_1} such that $\mathfrak{P}_i + J_{s1} = A_{s_1}$. We pick maximal ideals $\mathfrak{M}_1, \ldots, \mathfrak{M}_k$ in spec (A_{s_1}) such that $\mathfrak{P}_i \subseteq \mathfrak{M}_i$ for $i = 1$ to k and let $\mathfrak{M}_0 = \mathfrak{M}_1 \cap \cdots \cap \mathfrak{M}_k$. Then $J_{0_{s_1}} + \mathfrak{M}_0 = A_{s_1}$. Let $a + b = 1$ for a in J_{s1}^2 and b in $\mathfrak{M}_0$. Let $f_r' = bf_r + a$. It follows that $J_{0_{s_1}} = (f_1, \ldots, f_{r-1}, f_r') + J_{0_{s_1}}^2 \mathfrak{M}$.

Hence there is $s_2 = 1 + t_2$ in $1 + J_{0_{s_1}} \mathfrak{M}_0$, such that $J_{0_{s_1 s_2}} = (f_1, \ldots, f_r')$. Clearly, s_2 is not in $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_r$. If $\mathfrak{P}$ is any other associated prime of A_{s_1} and $s_2 = 1 + t_2$ is in $\mathfrak{P}$, then $J_{0_{s_1}} + \mathfrak{P} = A_{s_1}$, which is impossible. So, we have found a nonzero divisor s_2 in $1 + J_{0_{s_1}} \mathfrak{M}_0$ such that $J_{0_{s_1 s_2}} = (f_1, f_2, \ldots, f_r')$.

Now let K be the kernel of $\varphi_{s_1 s_2} : Q_{s_1 s_2} \to J_{s_1 s_2}$. Since $J_{s_1 s_2} = (f_1, f_2, \ldots, f_{r-1}, f_r'^{(r-1)!})$, there are $e_1', e_2', \ldots, e_r'$ in $Q_{s_1 s_2}$ such that $\varphi(e_1') = f_1, \ldots, \varphi(e_{r-1}') = f_{r-1}, \varphi(e_r') = f_r'^{(r-1)!}$.

By tensoring $0 \to K \to Q_{s_1 s_2} \to J_{s_1 s_2} \to 0$ by $A_{s_1 s_2}/\mathfrak{M}_{0 s_2}$, we get an exact sequence $0 \to K/\mathfrak{M}_0 K \to Q_{s_1 s_2}/\mathfrak{M}_0 Q_{s_1 s_2} \xrightarrow{\bar{\varphi}} J_{s_1 s_2}/J_{s_1 s_2}\mathfrak{M}_0 \approx A_{s_1 s_2}/\mathfrak{M}_0 s_2 \to 0$ and $\overline{\varphi(e_r')} = \overline{f^{(r-1)!}}$ is a unit in $A_{s_1 s_2}/\mathfrak{M}_{0_{s_2}}$. (Bar means module $\mathfrak{M}_{0_{s_2}}$). So there are $E_1, E_2, \ldots, E_{r-1}$ in K, such that images of $E_1, E_2, \ldots, E_{r-1}, e'_r$ is a basis of $Q_{s_1 s_2}/\mathfrak{M}_0 Q_{s_1 s_2}$.

Write $e_1 = be'_1 + aE_1, e_2 = be'_2 + aE_2, \ldots, e_{r-1} = be'_{r-1} + aE_{r-1}, e_r = e'_r$. It is easy to see that $e_1, \ldots, e_r$ is a basis of $Q_{s_1 s_2 W}$, where $W = 1 + J_{s_1 s_2}\mathfrak{M}_0$. So, there is $s_3 = 1 + t_3$ in $1 + J_{s_1 s_2}\mathfrak{M}_0$ such that $e_1, \ldots, e_r$ is a basis of $Q_{s_1 s_2 s_3}$. As before, s_3 is a nonzero divisor in $A_{s_1 s_2}$. Of course, $\varphi_{s_1 s_2 s_3}(e_1) = bf_1, \ldots, \varphi_{s_1 s_2 s_3}(e_{r-1}) = bf_{r-1}, \varphi_{s_1 s_2 s_3}(e_r) = (f'_r)^{(r-1)!}$. By further inverting a nonzero divisor in $1 + J_{0 s_1 s_2}$, we can also assume that $bf_1, \ldots, bf_{r-1}, f'_r$ generate $J_{0 s_1 s_2 s_3}$.

So, we are able to find a nonzero divisor s in A and a free basis $e_1, e_2, \ldots, e_r$ of Q_s such that, after replacing f_1 by $bf_1, \ldots, f_{r-1}$ by bf_{r-1} and f_r by f'_r, we have

(1) s is in I and $su + t = 1$ for some t in J_0 and u in A.

(2) $sJ_0 \subseteq (f_1, \ldots, f_r)$

(3) $\varphi(e_1) = f_1, \ldots \varphi(e_{r-1}) = f_{r-1}, \varphi(e_r) = f_r^{(r-1)!}$.

We had to go through all these technicalities because we wanted to have a nonzero divisor s. Now let $st = g_1 f_1 + g_2 f_2 + \cdots + g_r f_r$ and let $s^k Q \subseteq \bigoplus_{i=1}^{r} Ae_i \approx A^r$ for some $k \geqslant 0$.

By replacing Q by $s^k Q$ and I by $s^k I$, we can assume that

(4) s^{k+1} is in I,

(5) $ts = g_1 f_1 + \cdots + g_r f_r$.

(6) There is an inclusion $i : Q \to A^r = Ae_1 + \cdots + Ae_r$ such that $Q_s = A^r_s$ and

(7) $\varphi_s(e_i) = f_i$ for $i = 1$ to $r - 1$ and $\varphi_s(e_r) = f_r^{(r-1)!}$.

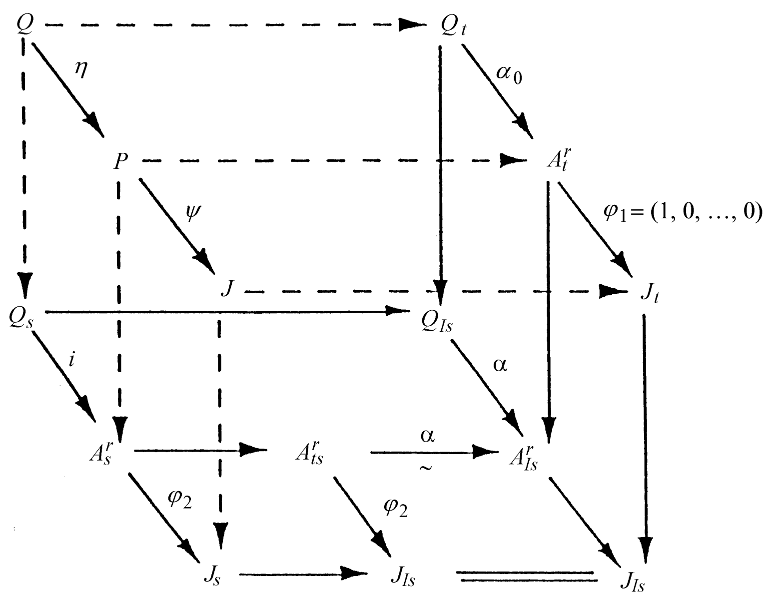
Let $A_r = A_r(\mathbb{Z})$ be as in Sect. 2 and let us consider the map $A_r \to A$ that sends X_i to f_i, Y_i to g_i for $i = 1$ to r and T to t, S to s, U to u and V to 1. By the theorem of Suslin ([S]) there is an $r \times r$ matrix γ in $\mathbb{M}_r(A_r)$ with its first column equal to the transpose of $(X_1, X_2, \ldots, X_{r-1}, X_r^{(r-1)!})$ and with $\det(\gamma) = (ST)^a$ in A_r, for some integer $a \geqslant 1$. Now let α be the image of γ in $\mathbb{M}_r(A)$.

We shall consider α as a map $\alpha : A^r \to A^r$ and let $\alpha_0 : Q \to A^r$ be the restriction of α to Q.

Define the A-linear map $\varphi_0 : A^r \to A$ such that $\varphi_0(e_i) = f_i$ for $i = 1$to $r - 1$ and $\varphi_0(e_r) = f_r^{(r-1)!}$. Also let $\varphi_1 = (1, 0, \ldots, 0) : A^r_t \to A_t$ be the map defined by $\varphi_1(e_1) = 1$ and $\varphi_1(e_i) = 0$ for $i = 2$ to r. Also let $\varphi_2 = (\varphi_0)_s$. Note that $\varphi : Q \to IJ$ is the restriction of φ_0 to Q and hence the diagram

$$\begin{tikzcd} Q_t \arrow[r, "\varphi"] \arrow[d, "\alpha_0"] & IJ_t \arrow[d] \\ A_t^r \arrow[r, "\varphi_1"] & A_t \end{tikzcd}$$

is commutative.
Now consider the following fibre product diagram:



Here $\varphi_2 = (\varphi_0)_s$ is a surjective and the maps η and ψ on the upper left hand corner are given by the properties of fibre product diagram.

Clearly, the map $\psi : P \to J$ is surjective. Further, since α is the image of γ it follows from (2.6) that $[P] - [A^r] =$ image of $-\lambda_r = -[A/J_0]$.

Now it remains to show that $Q \oplus A^r$ maps onto $I \oplus P$.

Note that the diagram

$$\begin{tikzcd} Q \arrow[r, "\phi"] \arrow[d, "\eta"] & IJ \arrow[d] \\ P \arrow[r] & J \end{tikzcd}$$

is commutative because it is so on $D(t)$ and $D(s)$. Also note that the map $\eta_s : Q_s \to P_s$ is an isomorphism and hence $s^p P$ is contained in $\eta(Q)$ for some $p \geqslant 1$.

Write $K = \text{kernel}(\psi)$. So, the sequence $0 \to K \to P \overset{\psi}{\to}\to J \to 0$ is exact.

Since $Tor_i(J, A/s^pA) = 0$, the sequence $0 \to K/s^pK \to P/s^pP \to J/s^pJ \approx A/s^pA \to 0$ is exact. In the following commutative diagram of exact sequences

$$\begin{array}{ccc} P/s^pP & \xrightarrow{\bar{\psi}} & J/s^pJ \approx A/s^pA \to 0 \\ \downarrow & & \downarrow \\ A_t^r/s^pA_t^r & \xrightarrow{\bar{\varphi}_1} & J_t/s^pJ_t \to 0, \end{array}$$

the vertical maps are isomorphism. But since $\bar{\varphi}_1 = (1, 0, \ldots, 0)$, $K/s^pK =$ kernel $\bar{\psi} \approx$ ker $\bar{\varphi}_1$ is a free A/s^pA-module of rank $r - 1$.

Now write $M = \text{kernel } \phi$. Then we have the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & K & \to & P & \overset{\psi}{\to}\longrightarrow & J & \to & 0 \\ & & \uparrow & & \uparrow \eta & & \uparrow & & \\ 0 & \to & M & \to & Q & \longrightarrow & IJ & \to & 0 \end{array}$$

of exact sequences.

Define the map $\delta : P \oplus I \to J + I = A \to 0$ such that $\delta(p, x) = \psi(p) - x$ for p in P and x in I and let $L = \text{kernel}(\delta)$. So, $0 \to L \to P \oplus I \to J + I = A \to 0$ is an exact sequence and $L \oplus A$ is isomorphic to $P \oplus I$. So, it is enough to show that $Q \oplus A^{r-1}$ maps onto L.

But L is isomorphic to $\psi^{-1}(IJ)$ and we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \to & K & \to & L & \overset{\psi}{\to} & J & \to & 0 \\ & & \uparrow & & \uparrow \eta & & || & & \\ 0 & \to & M & \to & Q & \longrightarrow & IJ & \to & 0 \end{array}$$

Note that s^pK is contained in $\eta(M)$. So K/s^pK maps onto $K/\eta(M)$. Therefore $K/\eta(M)$ is generated by $r - 1$ elements.

As $Q \oplus K/\eta(M)$ maps onto $L, Q \oplus A^{r-1}$ maps onto L. This completes the proof of Theorem 3.1.

5. The theorem of Murthy

In this section we give some applications of (3.1), which was inspired by the fact that the Picard group of smooth curves over algebraically closed fields are divisible.

Theorem 5.1. *Let A be a commutative ring of dimension n and I be a locally complete intersection ideal of height n in A. Suppose that I contains a locally complete intersection ideal J' of height $n-1$ and there is a projective A-module Q_0 of rank $n-1$ that maps onto J'. If the image of I in A/J' is invertible and is divisible by $(n-1)!$ in $\mathrm{Pic}(A/J')$, then there is a projective A-module Q of rank n that maps onto I and $(n-1)!([Q]-[Q_0]-[A]) = -[A/I]$ in $K_0(A)$.*

Proof. Let bar "–" denote images in A/J'. Since $\bar{I}$ is divisible by $(n-1)!$ in $\mathrm{Pic}(A/J')$, its inverse is also divisible by $(n-1)!$. Let J be an ideal of A such that $J' \subseteq J, I+J = A$ and $\bar{J}^{(n-1)!} = \bar{I}^{-1}$ in $\mathrm{Pic}(A/J')$. Hence $\overline{IJ}^{(n-1)!} = (J', f)/J'$ for some f in I.

Write $G = J' + J^{(n-1)!}$. We can also find a g in J such that $J = (J', g) + J^2$. Since $J'/J'J$ is locally generated by $(n-1)$ elements, $J'/J'J$ is $(n-1)$-generated. Let $g_1, \ldots, g_{n-1}$ generate $J'/J'J$. We can find an element s in J, such that $J'_{1+s} = (g_1, \ldots, g_{n-1})$ and $J_{1+s} = (g_1, \ldots, g_{n-1}, g)$. Hence it also follows that $G_{1+s} = (g_1, \ldots, g_{n-1}, g^{(n-1)!})$. Therefore $G = (g_1, \ldots, g_{n-1}) + J^{(n-1)!}$ and $J = (g_1, \ldots, g_{n-1}, g) + J^2$. Since $\overline{IJ}^{(n-1)!} = (J', f)/J'$, it follows that $IG = (f, J')$. As $Q_0 \oplus A$ maps onto IG, by Theorem 3.1, there is a surjective map $\varphi : Q_0 \oplus A^{n+1} \to I \oplus P$, where P is a projective A-module of rank n with $[P]-[A^n] = -[A/J]$. Let $Q = \varphi^{-1}(I)$. Then Q maps onto I and $Q \oplus P \approx Q_0 \oplus A^{n+1}$. So, $[Q]-[Q_0]-[A] = -([P]-[A^n]) = [A/J]$. Hence $(n-1)!([Q]-[Q_0]-[A]) = (n-1)![A/J] = [A/G] = -[A/I]$. This completes the proof of (5.1)

Corollary 5.2. *Let A be a smooth affine domain over an infinite field k and let $X = \mathrm{Spec}\ A$. Assume that for all smooth curves C in X, $\mathrm{Pic}\ C$ is divisible by $(n-1)!$. If I is a smooth ideal of* height $n = \dim X$, *then there is a projective module Q of* rank n, *such that Q maps onto I and $(n-1)!([Q]-[A^n]) = -[A/I]$.*

Proof. We can find elements $f_1, \ldots f_{n-1}$ in I such that $C = \mathrm{Spec}(A/(f_1, \ldots, f_{n-1}))$ is smooth [Mu2, Corollary 2.4]. Now we can apply (5.1) with $Q_0 = A^{n-1}$.

Remark. Unless k is an algebraically closed field, there is no known example of affine smooth variety that satisfy the hypothesis of (5.2) about the divisibility of the Picard groups. When k is an algebraically closed field, (5.2) is a theorem of Murthy [Mu2, Theorem 3.3] .

Part Two: Sections 6–8

Projective modules and Chern classes

This part of the paper is devoted to construct Projective modules with certain cycles as the total Chern class and to consider related questions. Our main results in this Part are in Sect. 8.

6. Grothendieck γ-filtration and Chern class formalism

As mentioned in the introduction, for a noetherian scheme X, $K_0(X)$ will denote the Grothendieck group of locally free sheaves of finite rank over X. All schemes we consider are connected and has an ample line bundle on it.

In this section we shall recall some of the formalisms about the Gorthendieck $\gamma - filtrations$ of the Grothendieck groups and about Chern classes. The main sources of this material are [SGA6], [Mn] and [FL].

Definitions and Notations 6.1. Let X be noetherian scheme of dimension n and let $K_0(X)[[t]]$ be the power series ring over $K_0(X)$. Then

a) $\lambda_t = 1 + t\lambda^1 + t^2\lambda^2 + \cdots$ will denote the additive to multiplicative group homomorphism from $K_0(X)$ to $1 + tK_0(X)[[t]]$ induced by the exterior powers, that is $\lambda^i([E]) = [\Lambda^i(E)]$ for any locally free sheaf E of finite rank over X, and $i = 0, 1, 2, \ldots,$

b) $\gamma_t = 1 + t\gamma^1 + t^2\gamma^2 + \cdots$ will denote the map $\lambda_{t/1-t}$, which is also an additive to multiplicative group homomorphism.

c) We let $F^0K_0(X) = K_0(X)$,$F^1K_0(X) = Kernel(\epsilon)$ where $\epsilon : K_0(X) \to \mathbb{Z}$ is the rank map.

For positive integer $k, F^kK_0(X)$ will denote the subgroup of $K_0(X)$ generated by the elements $\gamma^{k_1}(x_1)\gamma^{k_2}(x_2)\ldots\gamma^{k_r}(x_r)$ such that $\sum_{i=1}^r k_i \geqslant k$ and x_i in $F^1K_0(X)$. We shall often write $F^i(X)$ for $F^iK_0(X)$.

Recall that

$$F^0(X) \supseteq F^1(X) \supseteq F^2(X) \supseteq \cdots$$

is the Grothendieck γ-filtration of $K_0(X)$. Also note that $F^{n+1}(X) = 0$ (see [FL,Mn]).

d) $\Gamma(X) = \bigoplus_{i=0}^n \Gamma^i(X)$ will denote the graded ring associated to the Grothendieck γ-filtration.

If x is in $F^k(X)$, then the image of x in $\Gamma^k(X)$ will be called the $cycle\ of\ x$ and be denoted by $Cycle(x)$.

e) For a locally free sheaf E of rank r over X and for nonnegative interger i, the ith Chern class of E is defined as

$$c_i(E) = \gamma^i([E] - r) \ \ modulo \ \ F^{i+1}(X).$$

This will induce a Chern class homomorphism

$$c_t : K_0(X) \to 1 + \bigoplus_{i=1}^{n} \Gamma^i(X)t^i$$

which is also an additive to multiplicative group homomorphism. We write

$$c_t(x) = 1 + \sum_{i=1}^{n} c_i(x)t^i$$

with $c_i(x)$ in $\Gamma^i(X)$.

f) We recall some of the properties of this Chern class homomorphism: (1) if x is in $F^k(X)$, then

$$c_i(x) = 0 \ \ for \ \ 1 \leqslant i < r$$

$$c_r(x) = (-1)^{r-1}(r-1)!Cycle(x),$$

(2) if E is a locally free sheaf of rank r and if E maps onto a locally complete intersection sheaf I of ideals of height r then $[\mathfrak{O}_X/I] = \sum_{i=0}^{r}(-1)^i\lambda^i[E]$ is in $F^r(X)$ and

$$c_r([E]) = (-1)^r Cycle([\mathfrak{O}_X/I])$$

Now we shall set up some notations about the formalism of Chern classes in Chow groups.

Notations and Facts 6.2. Let X be a noetherian scheme of dimension n and let

$$CH(X) = \bigoplus\nolimits_{i=0}^{n} CH^i(X)$$

be the Chow group of cycles of X modulo rational equivalence. Assume that X is nonsingular over a field. Then

a) There is a Chern class homomorphism

$$C_t : K_0(X) \to 1 + \bigoplus_{i=1}^{n} CH^i(X)t^i$$

which is an additive to multiplicative group homomorphism. We write $C_t(x) = 1 + \sum_{i=1}^{n} C_i(x)t^i$ with $C_i(x)$ in $CH^i(X)$.

b) For nonnegative integer $k, \mathfrak{F}^k K_0(X)$ or simply $\mathfrak{F}^k(X)$ will denote the subgroup of $K_0(X)$ generated by $[M]$, where M runs through all coherent sheaves on X with $codimension(support M)$ atleast k. For such a coherent sheaf $M, Cycle M$ will denote the codimention $r-cycle$ in the Chow group of X. (There will be no scope of confussion with notation $Cycle\ x$ we introduced in (6.1 c).)

c) We recall some of the properties of this Chern Class homomorphism (see [F]):
(1) if x is in $\mathfrak{F}^r K_0(X)$ then

$$C_i(x) = 0 \quad 1 \leqslant i < r \quad and$$
$$C_r(x) = (-1)^{r-1}(r-1)! Cycle(x)$$

(2) If E is a locally free sheaf of finite rank over X and there is a surjective map from E onto a locally complete intersection ideal sheaf I of height r then

$$C_r(E) = (-1)^r Cycle(\mathfrak{O}_X/I).$$

It is known that for a nonsingular variety X over a field, the $\gamma-filtration\ F^r(X)$ of $K_0(X)$ is finer than the filtration $\mathfrak{F}^r(X)$ i.e. $F^r(X) \subseteq \mathfrak{F}^r(X)$. Following is an example of a nonsingular affine ring over a field k, for which these two filtrations indeed disagree.

Example 6.3. Following the notations in Sect. 2, for a fixed positive integer n and a field k, let

$$A_n = A_n(k) = \frac{k[S, T, U, V, X_1, \dots, X_n, Y_1, \dots, Y_n]}{(SU + TV - 1, X_1Y_1 + \cdots + X_nY_n - ST)}$$

$$B_n = B_n(k) = \frac{k[T, X_1, \dots, X_n, Y_1, \dots, Y_n]}{(X_1Y_1 + \cdots + X_nY_n - T(1+T))}$$

Then for $X = Spec A_n$ or $Spec B_n$,

$$\mathfrak{F}^r(X) \approx \mathbb{Z} \ \ for \ \ 1 \leqslant r \leqslant n \ \ and$$
$$\mathfrak{F}^r(X) = 0 \ \ for \ \ n < r.$$

Further,

$$F^n(X) = (n-1)! \mathfrak{F}^n(X) \quad and \quad F^r(X) = 0 \quad for \quad n < r.$$

Proof. The computation of $\mathfrak{F}^r(X)$ is done exactly as in [Sw1],(see (2.1) in case of A_n). Since $F^r(X)$ is contained in $\mathfrak{F}^r(X)$, $F^r(X) = 0$ for $r > n$.

For definiteness, let X be $SpecA_n$. So, $\lambda_n = [A_n/(X_1, \dots, X_n, T)]$ is the generator of $\mathfrak{F}^r(X)$ for $1 \leqslant r \leqslant n$. Also λ_n is the generator of $F^1(X) = \mathfrak{F}^1(X)$. By (2.6) in , there is a projective $A-module\ P$ of rank n with $[P] - n = -\lambda_n$ so that P maps onto the ideal $J = (X_1, \dots, X_{n-1}) + I^{(n-1)!}$, where I is the ideal $(X_1, \dots, X_n, T)A_n$. Hence $(n-1)!\lambda_n = -[A_n/J]$,(see (7.3)), is in $F^n(X)$. Also since $\mathfrak{F}^{n+1}(X) = 0$, $\lambda_n^2 = 0$. Hence for $1 \leqslant k$,γ^k acts as a group homomorphism on $F^1(X)$. So,$F^n(X) = \mathbb{Z}\gamma^n(\lambda_n)$. As $F^{n+1}(X) = 0$, by (6.1), $\gamma_n((n-1)!\lambda_n) = c_n((n-1)!\lambda_n) = (-1)^{n-1}(n-1)!^2\lambda_n$. Hence $F^n(X) = \mathbb{Z}(n-1)!\lambda_n$. So the proof of (6.3) is complete.

7. Some more preliminaries

Following theorem (7.1) gives the Chern classes of the projective module P that we constructed in theorem (3.1).

Theorem 7.1. *Under the set up and notations of theorem (3.1), we further have*

$$c_i(P) = 0 \quad for \quad 1 \leqslant i < r \quad in \quad \Gamma^i(X) \bigotimes \mathbb{Q},$$
$$c_r(P) = (-1)^r Cycle(A/J) \quad in \quad \Gamma^r(X).$$

If X is nonsingular over a field then

$$C_i(P) = 0 \quad for \quad 1 \leqslant i < r \quad in \quad CH^i(X) \quad and$$
$$C_r(P) = (-1)^r Cycle(A/J) \quad in \quad CH^r(X).$$

Proof. Comments about Chern classes in $\Gamma(X)$ follow from (6.1), because $(r-1)!([P]-r) = -[A/J]$ is in $F^r(X)$. Similarly, since $[A/J_0]$ is in $\mathfrak{F}^r(X)$, the comments about Chern classes in Chow group follow from (6.2).

Remark 7.2. For historical reasons we go back to the statement of theorem 3.1. Let J_0 be an ideal in a noetherian commutative ring A and let $J_0 = (f_1, \dots, f_r) + J_0^2$.The part(i) of theorem (3.1) evolved in two stages.First, Boratynski [B] defined $J = (f_1, \dots, f_{r-1}) + J_0^{(r-1)!}$ and proved that there is a projective $A-module\ P$ of rank r that maps onto J.Then, Murthy[Mu2] added that if J_0 is a locally complete intersection ideal of height r then there is one such projective $A - module\ P$ of rank r, with $[P] - r = -[A/J_0]$, that maps onto J.

We shall be much concerned with such ideals J constructed, as above, from ideals J_0. Following are some comments about such ideals.

Natations and Facts 7.3. *For an ideal J in a Cohen-Macaulay ring A with $J = (f_1, \dots, f_r) + J^2$.We use the notation*

$$B(J) = B(J, f_1, \dots, f_r) = (f_1, \dots, f_{r-1}) + J^{(r-1)!}.$$

Then, we have

(1) $\sqrt{J} = \sqrt{B(J)}$,

(2) J is locally complete intersection ideal of height r if and only if so is $B(J)$.

(3) If J is locally complete intersection ideal of height r then $[A/B(J)] = (r-1)![A/J]$ in $K_0(X)$.

Proof. The proof of (1) is obvious.To see (2), note that locally, J_0 is generated by $f_1, \dots, f_r$ and $B(J_0)$ is generated by $f_1, \dots, f_{r-1}, f_r^{(r-1)!}$. To prove (3), let $J_k = (f_1, \dots, f_{(r-1)}) + J_0^k$ for positive integers k. Note that $0 \to J_k/J_{k+1} \to A/J_{k+1} \to A/J_k \to 0$ is exact and $A/J_0 \approx J_k/J_{k+1}$. Now (3) follows by induction and hence the proof of (7.3) is complete .

The following lemma describes such ideals $B(J_0)$ very precisely.

Lemma 7.4. *Let A be a Cohen-Macaulay ring and J be an ideal in A. Then $J = B(J_0) = B(J_0, f_1, \dots, f_r)$ for some ideal $J_0 = (f_1, \dots, f_r) + J_0^2$ if and only if $J = (f_1, \dots, f_{r-1}, f_r^{(r-1)!}) + J^2$.*

Proof. To see the direct implication, let $J = (f_1, \dots, f_{r-1}) + J_0^{(r-1)!}$ where $J_0 = (f_1, \dots, f_r) + J_0^2$. Then, it is easy to check that $J = (f_1, \dots, f_{r-1}, f_r^{(r-1)!}) + J^2$. Conversely, let $J = (f_1, \dots, f_{r-1}, f_r^{(r-1)!}) + J^2$. By Nakayama's lemma, there is an s in J such that

$$(1+s)J \subseteq (f_1, \dots, f_{r-1}, f_r^{(r-1)!}) \ \ and \ \ J = (f_1, \dots, f_{r-1}, f_r^{(r-1)!}, s).$$

Now we let $J_0 = (f_1, \dots, f_r, s)$. It follows that $J = B(J_0)$ and the proof of (7.4) is complete.

The following lemma will be useful in the next section.

Lemma 7.5. *Let A be a Cohen-Macaulay ring of dimension n and let I and J be two locally complete intersection ideals of height r with $I + J = A$.Then, if $IJ = B(\mathfrak{J})$ for some locally complete intersection ideal $\mathfrak{J}$ of height r then $I = B(I_0)$ for some locally complete intersection ideal I_0 of height r. Also* *if $I = B(I_0)$ and $J = B(J_0)$ for locally complete intersection ideals I_0, J_0 then $IJ = B(\mathfrak{J})$ for some locally complete intersection ideal $\mathfrak{J}$ of height r.*

The proof is straightforward.

Remark. With careful formulation of the statements, the Cohen-Macaulay condition in (7.3), (7.4), (7.5) can be dropped.

8. Results on Chern classes

Our approach here is that if Q is a projective module of rank r over a noetherian commutative ring A then we try to construct a projective $A-module$ Q_0 of rank $r-1$, so that the first $r-1$ Chern classes of Q are same as that of Q_0.Conversely, given a projective $A-module$ Q_0 of rank $r-1$ and a locally complete intersection ideal I of height r, we attempt to construct a projective $A-module$ Q of rank r such that the first $r-1$ Chern classes of Q and Q_0 are same and the top Chern class of Q is $(-1)^r Cycle([A/I])$. Our first theorem (8.1) suggests that if $r = n = dimX$, then for such a possibility to work, it is important that $[A/I]$ is divisible by $(n-1)!$.

Theorem 8.1. *Let A be noetherian commutative ring of dimension n and $X = SpecA$. Assume that $K_0(X)$ has no $(n-1)!$ torsion. Suppose that Q is a projective $A-module$ of rank n and Q_0 is a projective $A-module$ of rank $n-1$. Assume that the first $n-1$ Chern classes,in $\Gamma(X)$ (respectively, in $CH(X)$, if X is nonsingular over a field), of Q and Q_0 are same. Then $\sum_{i=0}^{n}(-1)^i[\Lambda^i Q]$ is divisible by $(n-1)!$ in $K_0(X)$. That means that if Q maps onto a locally complete intersection ideal I of height n, then $[A/I]$ is divisible by $(n-1)!$ in $K_0(X)$.*

Proof. We can find a projective $A-module$ P of rank n such that $Q \bigoplus A^n \approx Q_0 \bigoplus A \bigoplus P$.It follows that $c_i(P) = 0$ for $1 \leqslant i < n$ and $c_n(P) = c_n(Q)$ in $\Gamma(X)$. Write $\rho = [P] - n$.

We claim that for $r = 0$ to $n-1$, $\beta_r \rho$ is in $F^{r+1}(X)$, where $\beta_r = \Pi_{i=1}^{r-1}(i!)$. By Induction, assume that $\beta_{r-1}\rho$ is in $F^r(X)$. Since $c_r(\beta_r\rho) = \beta_r c_r(\rho) = 0$, also since $c_r(\beta_r\rho) = (-1)^{r-1}(r-1)!\beta_r\rho$ the claim follows.

So, $\beta_{n-1}\rho$ is in $F^n(X)$. Hence $c_n(\beta_{n-1}\rho) = (-1)^{(n-1)}(n-1)!\beta_{n-1}\rho$. Since $K_0(X)$ has no β_{n-1} torsion, it follows that $c_n(\rho) = (-1)^{n-1}(n-1)!\rho$.Since $c_n(\rho) = c_n(Q) = (-1)^n \sum_{i=0}^{n}(-1)^i[\Lambda^i Q]$, the theorem follows.

We argue similarly when X is nonsingular over a field and Chern classes take values in the Chow gorup. In this case, we use (6.2c). This completes the proof of (8.1).

Our next theorem (8.2) is a converse of (8.1).

Theorem 8.2. *Let A be a noetherian commutative ring of dimension n and $X = SpecA$. Let J be a locally complete intersection ideal of height $r > 0$ with $J = (f_1, \dots, f_{r-1}, f_r^{(r-1)!}) + J^2$ (hence $J = B(J_0)$ for some locally complete intersection ideal J_0 of height r). Let Q be a projective $A-module$ of rank r that maps onto J. Then there is a projective $A - module$ Q_0 of rank $r-1$ such that,*

(1) $[Q_0 \bigoplus A] = [Q] + [A/J_0] \quad in \quad K_0(X)$,

(2)

$$c_i(Q_0) = c_i(Q) \quad in \quad \Gamma(X) \bigotimes \mathbb{Q} \quad for \quad 1 \leqslant i < r \quad and$$

if X is nonsingular over a field then

$$C_i(Q_0) = C_i(Q) \quad in \quad CH(X) \quad for \quad 1 \leqslant i < r.$$

(3) If $K_0(X)$ has no torsion (respectively, no $(n-1)!$ torsion,in case X is nonsingular over a field) then such a $[Q_0]$ satisfying (2) is unique in $K_0(X)$.

Proof. By (3.1) with $I = A$, there is a surjective map

$$\psi : Q \bigoplus A^r \to A \bigoplus P$$

where P is a projective $A-module$ of rank r that maps onto J and $[P] - r = -[A/J_0]$. We let $Q_0 = kernel\,(\psi)$. Then $Q_0 \bigoplus A \bigoplus P \approx Q \bigoplus A^r$. This settles (1). By (7.1), first $r-1$ Chern classes of P, in $\Gamma(X) \bigotimes \mathbb{Q}$, are zero. Hence it follows that $c_i(Q) = c_i(Q_0)$ for $1 \leqslant i < r$. In case X is nonsingular, the argument runs similarly. So, the proof of (2) of (8.2) is complete.

To prove (3), let Q' be another projective $A - module$ of rank $r-1$ satisfying (2) and let $\rho = ([Q_0] - [Q'])$. Since the total Chern classes of Q_0 and Q' in $\Gamma(X) \bigotimes \mathbb{Q}$ (respectively, in $CH(X)$, in case X is nonsingular), are same,the total Chern class $c(\rho) = 1$ in the respective groups.

For a positive integer r let $\beta_r = \Pi_{i=1}^{r-1}(i!)$. By induction, as in (8.1), it follows that $\beta_n \rho$ is in $F^{n+1}(X) \bigotimes \mathbb{Q} = 0$ (respectively, in $\mathfrak{F}_{n+1}(X) = 0$). Hence the proof of (8.2) is complete.

Following theorem (8.3) gives a construction of projective modules with certain given cycles as its total Chern class.

Theorem 8.3. *Let A be a Cohen- Macaulay ring of dimension n and $X = SpecA$. Let r_0 be an integer with $2r_0 \geqslant n$.*

Let Q_0 be a projective $A-module$ of rank $r_0 - 1$, such that for all locally *complete intersection subschemes Y of X with $codimension\ Y \geqslant r_0$, the restriction $Q_0|Y$ of Q_0 to Y is trivial. Also let r be another integer with $r_0 \leqslant r \leqslant n$ and for $k = r_0$ to r, let I_k be locally complete intersection ideals of height k, with $I_k = (f_1, \ldots, f_{k-1}, f_k^{(k-1)!}) + I_k^2$ (hence $I_k = B(I_{k0})$, for some locally complete intersection ideal I_{k0} of height k).*

Then there is a projective $A-module$ Q_r of rank r such that

(1) Q_r maps onto I_r,

(2)

$$[Q_r] - r = ([Q_0] - (r_0 - 1)) + [A/J_{r_0}] + \cdots + [A/J_r],$$

where J_k is a locally complete intersection ideal of height k such that $(k-1)![A/J_k] = -[A/I_k]$ and further, $[P_k] - k = -[A/J_k]$ for some projective $A-module$ P_k of rank k, for $r_0 \leqslant k \leqslant r$.

(3)

$$c_k(Q_r) = c_k(Q_0) \quad in \quad \Gamma^k(X) \bigotimes \mathbb{Q} \quad for \quad 1 \leqslant k < r_o$$

$$c_k(Q_r) = (-1)^k Cycle([A/I_k]) \quad in \quad \Gamma^k(X) \bigotimes \mathbb{Q} \quad for \quad r_o \leqslant k \leqslant r.$$

If X is nonsingular over a field,then

$$C_k(Q_r) = C_k(Q_0) \quad in \quad CH^k(X) \quad for \quad 1 \leqslant k < \ r_0 \quad and$$

$$C_k(Q_r) = (-1)^k Cycle(A/I_k) \quad in \quad CH^k(X) \quad for \quad r_0 \leqslant k \leqslant r.$$

Caution. In the statement of (8.3) the generators $f_1, \ldots, f_{k-1}, f_k^{(k-1)!}$ of I_k/I_k^2 depend on k.

Remark 8.4. A free $A-module$ Q_0 of rank $r_0 - 1$ will satisfy the hypothesis of (8.3). If $r_0 = n-1$, then any projective $A-module$ Q_0 of rank $r_0 - 1$ with trivial determinant will also satisfy the hypothesis of (8.3).

The proof of (8.3) follows, by induction, from the following proposition (8.5).

Proposition 8.5. *Let A be a Cohen-Macaulay ring of dimension n and $X = SpecA$ and let r be a positive integer with $2r \geqslant n$ and $r \leqslant n$. Let Q_0 be a projective $A-module$ of rank $r-1$ such that for any locally complete intersection closed subscheme Y of codimension atleast r, the restriction $Q_0|Y$ of Q_0 to Y is trivial. Also let I be a locally complete intersection ideal of height r with $I = (f_1, \ldots, f_{r-1}, f_r^{(r-1)!}) + I^2$ (hence $I = B(I_0)$ for some locally complete intersection ideal I_0 of height r).*

Then there is a projective $A-module$ Q of rank r such that

(1) Q maps onto I,

(2)

$$[Q] - r = ([Q_0] - (r-1)) + [A/J_0],$$

where J_0 is a locally complete intersection ideal of height r such that $(r-1)![A/J_0] = -[A/I]$ and further there is a projective $A - module$ P of rank r such that $[P] - r = -[A/J_0]$.

(3)

$$c_k(Q) = c_k(Q_0) \quad in \quad \Gamma^k(X) \bigotimes \mathbb{Q} \quad for \quad 1 \leqslant k < r, \quad and$$

$$c_k(Q) = (-1)^r Cycle([A/I]) \quad in \quad \Gamma^k(X) \bigotimes \mathbb{Q} \quad for \quad k = r.$$

If X is nonsingular over a field, then

$$C_k(Q) = C_k(Q_0) \quad in \quad CH^k(X) \quad for \quad 1 \leqslant k < r$$

and

$$C_k(Q) = (-1)^r Cycle(A/I) \quad in \quad CH^k(X) \quad for \quad k = r.$$

(4) For any locally complete intersection closed subscheme Y of X of codimension atleast $r+1$, the restriction $Q|Y$ is trivial.

Before we prove (8.5), we state the following proposition from [CM].

Proposition 8.6. *Let A be a noetherian commutative ring and J be a locally complete intersection ideal of height r with J/J^2 free. Suppose I is an ideal with $dim A/I < r$. If $\pi : K_0(A) \to K_0(A/I)$ is the natural map then $\pi([A/J]) = 0$.*

The proof is done by finding a locally complete intersection ideal J' of height r such that $J' + I = A = J' + J$ and $J \cap J'$ is complete intersection. Now it follows that $\pi([A/J]) = -\pi([A/J']) = 0$.

Proof of (8.5). We have

$$I = (f_1, \ldots, f_{r-1}, f_r^{(r-1)!}) + I^2 = B(I_0) = (f_1, \ldots, f_{r-1}) + I_0^{(r-1)!},$$

where I_0 is a locally complete intersection ideal of height r with $I_0 = (f_1, \ldots, f_r) + I_0^2$. We can also assume that $f_1, \ldots, f_r$ is a regular sequence. By hypothesis Q_0/IQ_0 is free of rank $r-1$. Let $e_1, \ldots, e_{r-1}$ be elements in Q_0 whose images forms a basis of Q_0/IQ_0.So, there is a map $\phi_0 : Q_0 \to I$ such that $\phi_0(e_i) - f_i$ is in I^2 for $i = 1$ to $r-1$.

So, $(\phi_0(Q_0), f_r^{(r-1)!}) + I^2 = I$. By Nakayama's lemma there is an s in I such that

$$(1+s)I \subseteq (\phi_0(Q_0), f_r^{(r-1)!}) \quad and \quad I = (\phi_0(Q_0), f_r^{(r-1)!}, s).$$

Let Q_0^* be the dual of Q_0. Then (ϕ_0, s^2) is basic in $Q_0^* \bigoplus A$ on the set $\mathfrak{P} = \{\wp \ in \ SpecA \ : \ f_r \ is \ in \ \wp \ and \ height \ (\wp) \leqslant r-1\}$. There is a generalised dimension function $d \ : \ \mathfrak{P} \to \{0, 1, 2, \dots\}$ so that $d(\wp) \leqslant r-2$ for all $\wp$ in $\mathfrak{P}$ (see [P]). Since rank $Q_0^* = r-1 > d(\wp)$ for all $\wp$ in $\mathfrak{P}$, there is an h in Q_0^* such that $\phi = \phi_0 + s^2 h$ is basic in Q_0^* on $\mathfrak{P}$.

Write $\mathfrak{I} = (\phi(Q_0), f_r^{(r-1)!})$. It follows that (1) $\mathfrak{I}$ is a locally complete intersection ideal of height r, (2) $[A/\mathfrak{I}] = 0$, (3) $\mathfrak{I} + I^2 = I$ (4) $\mathfrak{I} = (g_1, \dots, g_{r-1}, f_r^{(r-1)!}) + \mathfrak{I}^2$ for some $g_1, \dots, g_{r-1}$ in $\mathfrak{I}$.

To see (1), note that $\mathfrak{I}$ is locally r generated and also since ϕ is basic in Q_0^* on $\mathfrak{P}$, $\mathfrak{I}$ has height r.Now since A is Cohen-Macaulay, $\mathfrak{I}$ is a locally complete intersection ideal of height r. Since

$$0 \to A/\phi(Q_0) \xrightarrow{f_r^{(r-1)!}} A/\phi(Q_0) \to A/\mathfrak{I} \to 0$$

is exact, (2) follows. Since $\phi = \phi_0 + s^2 h$,(3) follows. By hypothesis $Q_0/\mathfrak{I}Q_0$ is free of rank $r-1$ and hence (4) follows.

Because of (4), $\mathfrak{I} = B(\mathfrak{I}_0)$ for some locally complete intersection ideal $\mathfrak{I}_0$ of height r. From (3) it follows that $\mathfrak{I} = J \cap I$ for some locally complete intersection ideal J of height r and $I + J = A$. Since $\mathfrak{I} = B(\mathfrak{I}_0)$, by (7.5), $J = B(J_0)$ for some locally complete intersection ideal J_0 of height r.

Let $\phi : Q_0 \bigoplus A \to \mathfrak{I}$ be the surjective map $(\phi, f_r^{(r-1)!})$. We can apply theorem (3.1) and (7.1). There is a surjective map $\psi \ : \ Q_0 \bigoplus A^{r+1} \to P \bigoplus I$, where P is a projective $A-module$ of rank r that maps onto J and $[P] - r = -[A/J_0]$. Also

$$c_k(P) = 0 \quad for \quad 1 \leqslant k < r \quad in \quad \Gamma^k(X) \bigotimes \mathbb{Q},$$

$$c_r(P) = (-1)^r Cycle(A/J) \quad in \quad \Gamma^r(X).$$

If X is nonsingular over a field then

$$C_k(P) = 0 \quad for \quad 1 \leqslant k < r \quad in \quad CH^k(X) \quad and$$

$$C_r(P) = (-1)^r Cycle(A/J) \quad in \quad CH^r(X).$$

Now $Q = \psi^{-1}(I)$ will satisfy the assertions of the theorem.Clearly, Q maps onto I and (1) is satisfied. Note that $Q \bigoplus P \approx Q_0 \bigoplus A^{r+1}$ and hence

$$[Q] - r = ([Q_0] - (r-1)) - ([P] - r) = ([Q_0] - (r-1)) + [A/J_0].$$

Also $(r-1)![A/J_0] = [A/J] = -[A/I]$, since $[A/\mathfrak{J}] = 0$.This establishes (2).

Again since $Q \bigoplus P \approx Q_0 \bigoplus A^{r+1}$ and since the Chern classes of P are given as above, (3) follows.

To see (4), let Y be a locally complete intersection subscheme of X with codimension at least $r+1$.Let $\pi : K_0(X) \to K_0(Y)$ be the restriction map. Then $\pi([Q]-r) = \pi([Q_0]-(r-1)) + \pi([A/J_0]) = 0$ by (8.6). Hence the restriction $Q|Y$ is stably free.Since $r > dimY$, by cancellation theorem of Bass(see [EE]), $Q|Y$ is free. This completes the proof of (8.5).

Before we go into some of the applications let us recall(1.5, 1.6) that for a smooth affine variety $X = SpecA$ of dimension n over a field, $\mathfrak{F}^n(X) = F_0K_0(X) = \{[A/I] \ in \ K_0(X) : I$ is a locally complete intersection ideal of height $n\}$.

Following is an important corollary to theorem (8.3).

Corollary 8.7. *Suppose $X = SpecA$ is a smooth affine variety of dimension n over a field. Assume that $CH^n(X)$ is divisible by $(n-1)!$. Let Q_0 be projective $A - module$ of rank $n-1$ and x_n is a cycle in $CH^n(X)$. Then there is a projective $A - module$ Q of rank n such that*

$$C_i(Q) = C_i(Q_0) \quad for \quad 1 \leqslant i < n \quad and \quad C_n(Q) = x_n \quad in \quad CH^n(X).$$

Conversely, if Q is a projective $A - module$ of rank n, then there is a projective $A - module$ Q' of rank n such that

$$C_i(Q) = C_i(Q') \quad for \quad 1 \leqslant i < n \quad and \quad C_n(Q') = 0 \quad in \quad CH^n(X).$$

Proof. Since the Chern class map $C_n : \mathfrak{F}^n(X) \to CH^n(X)$ sends $[A/I]$ to $(-1)^{(n-1)}(n-1)!Cycle(A/I)$ (see [F]), this map is surjective. Since $\mathfrak{F}^n(X)$ $= F_0K_0(X)$, there is a locally complete intersection ideal I_0 of height n such that $C_n(A/I_0) = -x_n$. By theorem (8.3) with $I = B(I_0)$, there is a projective $A - module$ Q of rank n such that $[Q] - n = ([Q_0] - (n-1)) + [A/J]$ where J is a locally complete intersection ideal of height n with $(n-1)![A/J] = -[A/I] = -(n-1)![A/I_0]$. Hence

$$C_i(Q) = C_i(Q_0) \quad for \quad 1 \leqslant i < n \quad and$$

$$C_n(Q) = C_n([A/J]) = (-1)^{n-1}(n-1)!Cycle([A/J) =$$

$$(-1)^{n-1}Cycle((n-1)![A/J]) = (-1)^{n-1}Cycle(-(n-1)![A/I_0]) = x_n.$$

This establishes the direct implication.

To see the converse, note that, as above, there is a projective A-module P such that $C_i(P) = 0 \ for \ 1 \leqslant i < n$ and $C_n(P) = -C_n(Q)$. Now

$Q \bigoplus P \approx Q' \bigoplus A^n$ for some projective $A - module$ Q' of rank n. It is obvious that Q' satisfies the assertions. This completes the proof of (8.7).

Following theorem of Murthy ([Mu2]) follows from (8.7).

Theorem 8.8 (Murthy). *Let $X = SpecA$ be a smooth affine variety of dimension n over an algebraically closed field k. Let x_i be cycles in $CH^i(X)$ for $1 \leqslant i \leqslant n$. Then there is a projective $A - module$ Q_0 of rank $n - 1$ with the total Chern class $C(Q_0) = 1 + x_1 + \cdots + x_{n-1}$ in $CH(X)$ if and only if there is a projective $A - module$ Q of rank n with the toal Chern class $C(Q) = 1 + x_1 + \cdots + x_{n-1} + x_n$.*

Proof. In this case $CH^n(X)$ is divisible (see [Le,Sr,Mu2]). So the direct implication is immediate from (8.7).

To see the converse, let Q' be as in (8.7). Since $C_n(Q') = 0$,it follows from the theorem of Murthy([Mu2]) that $Q' \approx Q_0 \bigoplus A$ for some projective $A-module$ Q_0 of rank $n-1$. It is obvious that $C(Q_0) = 1+x_1+\cdots+x_{n-1}$. So the proof of (8.8) is complete.

Following is an alternative proof of the theorem of Mohan Kumar and Murthy ([MM]).

Theorem 8.9 ([MM]**).** *Let $X = SpecA$ be a smooth affine three fold over an algebraically closed field and let x_i be cycles in $CH^i(X)$ for $1 \leqslant i \leqslant 3$. Then*

(1) There is a projective $A-module$ Q_3 of rank 3 with total Chern class $C(Q_3) = 1 + x_1 + x_2 + x_3$,

(2)there is a projective $A - module$ Q_2 of rank 2 with total Chern class $C(Q_2) = 1 = x_1 + x_2$.

Proof. Because of (8.8), we need to prove (1) only.Let L be a line bundle on X with $C_1(L) = x_1$.We claim that there is a projective $A - module$ P of rank 3 so that $C_1(P) = 0$ and $C_2(P) = x_2$.Let $x_2 = (y_1 + \cdots + y_r) - (y_{r+1} + \cdots + y_s)$ where y_i is the cycle of A/I_i for prime ideals I_i of height 2 for $1 \leqslant i \leqslant s$.

For $1 \leqslant i \leqslant s$ there is an exact sequence

$$0 \to P_i \bigoplus G_i \to F_i \to A \to A/I_i \to 0$$

where F_i and G_i are free modules and P_i are projective $A - modules$ of rank 3.Since the total Chern classes $C(P_i) = C(A/I_i)$, it follows that $C_1(P_i) = C_1(A/I_i) = 0$ and $C_2(P_i) = C_2(A/I_i) = -Cycle(A/I_i) = -y_i$. There are free modules R and S and a projective $A-module$ P of rank 3 such that $P_1 \bigoplus \cdots P_r \bigoplus R \bigoplus P \approx P_{r+1} \bigoplus \cdots \bigoplus P_s \bigoplus S$. It follows that $C_1(P) = 0$ and $C_2(P) = (y_1 + \cdots + y_r) - (y_{r+1} + \cdots + y_s) = x_2$. This establishes the claim.

Let $P \bigoplus L \approx P' \bigoplus A$. Then $C_1(P') = C_1(L) = x_1$ and $C_2(P') = x_2$ and let $C_3(P') = z$ for some z in $CH^3(X)$. Again,by (8.7) there is a projective $A-module$ Q' of rank 3 such that the total Chern class $C(Q') = 1 + (x_3 - z)$. There is a projective $A - module$ Q_3 of rank 3 such that $Q' \bigoplus P' \approx Q_3 \bigoplus A^3$.We have, $C(Q')C(P') = C(Q_3)$. So the proof of (8.9) is complete.

The same proof of (8.9) yeilds the following stronger theorem (8.10).

Theorem 8.10. *Let $X = SpecA$ be a smooth affine three fold over any field k such that $CH^3(X)$ is divisible by two. Given x_i in $CH^i(X)$ for $1 \leqslant i \leqslant 3$, there is a Projective $A - module$ Q of rank 3 such that the total chern class $C(Q) = 1 + x_1 + x_2 + x_3$.*

Remark. For examples of smooth three folds that satisfy the hypothesis of (8.10) see [Mk2].

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