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Decomposition of Projective Modules

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Abstract. We study the decomposition of projective modules from K-theoratic point of view.

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1. Introduction

In [B], Boratynski proved the following theorem:

THEOREM 1.1. *Let A be a commutative ring and I be an ideal with $I = (f_1, \dots, f_n)+I^2$. Let $J = (f_1, \dots, f_{n-1})+I^{(n-1)!}$. Then J is image of a projective module P of rank n.*

This theorem of Boratynski had a far-reaching impact in the study of complete intersections in affine varieties (see [Ma1],[MK],[Mu], to mention a few). In this paper we give some applications of this theorem of Boratynski.

Mohan Kumar ([MK]) used it to prove the following theorem:

THEOREM 1.2. *Let A be a reduced affine ring of dimension n over an algebraically closed field k, and let Q be a projective A-module of rank n. Suppose that Q maps onto a complete intersection ideal $J = (f_1, \dots, f_n)$ of height n. Then $Q \approx Q_0 \oplus A$.*

One of the main results in this paper is the following theorem:

THEOREM 1.3. *Let A be a Noetherian commutative ring. Let $r_1, \dots, r_n$ be non-negative integers and $n \leqslant \dim(A)$. Let $f_1, \dots, f_n$ be a regular sequence in A. Suppose Q is a projective A-module of rank n. Suppose we have a surjective map $\varphi\colon Q \to (f_1^{r_1}, \dots, f_n^{r_n})$ and $r_1 r_2 \dots r_n$ is divisible by $(n-1)!$. Then $[Q] = [Q_0 \oplus A]$ in $K_0(A)$, for some projecive A-module Q_0.*

Further, if $n = \dim(A)$ is odd, A is Cohen–Macaulay and Q has trivial determinant, then $Q \approx Q_0 \oplus A$.

In fact, we give a more general theorem on such decomposition of projective modules as follows.

THEOREM 1.4. *Let A be a Noetherian commutative ring. Let $r_1, \dots, r_n$ be positive integers and $n \leqslant \dim(A)$. Let $J_0 = (f_1, \dots, f_n) + J_0^2$ be a locally complete intersection ideal of height n. Define*

$$J_1 = (f_2, \dots, f_n) + J_0^{r_1},$$
$$J_2 = (f_1^{r_1}, f_3, \dots, f_n) + J_1^{r_2},$$

and so on, and

$$J = J_n = (f_1^{r_1}, \dots, f_{n-1}^{r_{n-1}}) + J_{n-1}^{r_n}.$$

(Then $J = (f_1^{r_1}, \dots, f_n^{r_n}) + J^2$). Suppose that Q is a projective A-module of rank n and there is a surjective map $\varphi\colon Q \to J$. Assume that $(n-1)!$ divides $r_1 r_2 \dots r_n$. Then

$$[Q] = [Q_0 \oplus A] - \frac{r_1 \dots r_n}{(n-1)!}[A/J_0] \quad in \quad K_0(A),$$

for some projective A-module Q_0 of rank $n-1$.

These results in this paper are extensions of some of the results in [Ma1] and [Mu]. In this paper, A will always denote a commutative Noetherian ring and $K_0(A)$ will denote the Grothendieck Group of projective A-modules of finite rank.

2. Some Computations in the *K*-Groups

LEMMA 2.1. Let A be a Noetherian commutative ring and let J_0 be an ideal. Suppose $J_0 = (f_1, f_2, \dots, f_n) + J_0^2$ and $r_1, r_2, \dots, r_n$ be nonnegative integers. Define $J_1 = (f_2, f_3, \dots, f_n) + J_0^{r_1}$ and $J_2 = (f_1^{r_1}, f_3, \dots, f_n) + J_1^{r_2}$, and inductively define

$$J_k = (f_1^{r_1}, \dots, f_{k-1}^{r_{k-1}}, f_{k+1}, \dots, f_n) + J_{k-1}^{r_k}.$$

Then

$$J_n = (f_1^{r_1}, f_2^{r_2}, \dots, f_n^{r_n}) + J_n^2.$$

Proof. First, recall that for an ideal J, $J = (x_1, x_2, \dots, x_n) + J^2$ if and only if $J_t = (x_1, x_2, \dots, x_n)$ for some t with $J + At = A$.

We will prove, inductively, that

$$J_k = (f_1^{r_1}, f_2^{r-2}, \dots, f_k^{r_k}, f_{k+1}, \dots, f_n) + J_k^2.$$

It is enough to show the inductive step. So, we assume that

$$J_{k-1} = (f_1^{r_1}, \dots, f_{k-1}^{r_{k-1}}, f_k, \dots, f_n) + J_{k-1}^2.$$

Write $I = J_{k-1}$ and $J = J_k$. So, we have $I_t = (f_1^{r_1}, \dots, f_{k-1}^{r_{k-1}}, f_k, \dots, f_n)$ for some t with $I + At = A$. Therefore,

$$\begin{aligned} J_t &= (f_1^{r_1}, f_2^{r_2}, \dots, f_{k-1}^{r_{k-1}}, f_{k+1}, \dots, f_n) + I_t^{r_k} \\ &= (f_1^{r_1}, \dots, f_k^{r_k}, f_{k+1}, \dots, f_n). \end{aligned}$$

This finishes the proof. □

LEMMA 2.2. *Let A be a commutative Noetherian ring and J be an ideal. Suppose*

$$J = (f_1^{r_1}, f_2^{r_2}, \dots, f_n^{r_n}) + J^2$$

for some $f_1, f_2, \dots, f_n$ in A. Then there are ideals

$$J = J_n \subseteq J_{n-1} \subseteq J_{n-2} \subseteq \cdots \subseteq J_0,$$

where

$$J_k = (f_1^{r_1}, \dots, f_{k-1}^{r_{k-1}}, f_{k+1}, \dots, f_n) + J_{k-1}^{r_k}$$

for $k = 1, 2, \dots, n$ and

$$J_0 = (f_1, f_2, \dots, f_n) + J_0^2.$$

Proof. There is an element $s \in J$ such that $(1+s)J \subseteq (f_1^{r_1}, f_2^{r_2}, \dots, f_n^{r_n})$. Let $J_0 = (f_1, f_2, \dots, f_n, s)$. The rest of the lemma follows easily. □

LEMMA 2.3. *Let A be a noetherian commutative ring and I be an ideal of height n. Suppose that $I = (f_1, f_2, \dots, f_n) + I^2$ for some $f_1, f_2, \dots, f_n$ and let $J = (f_1, f_2, \dots, f_{n-1}) + I^r$ for some positive integer r. If I is locally complete intersection ideal then J is also a locally complete intersection ideal and $[A/J] = r[A/I]$ in $K_0(A)$.*

Conversely, if J is a locally complete intersection ideal then so is I.

Proof. First, assume that I is a locally complete intersection ideal of height n. We will find a regular sequence $g_1, \dots, g_n$ such that $g_i - f_i \in I^r$. We will do this by induction. Suppose we have already picked a regular sequence $g_1, \dots, g_k$ such that $g_i - f_i \in I^r$ for $i = 1$ to k and $k < n$. Note that $I = (g_1, \dots, g_k, f_{k+1}, \dots, f_n) + I^2$. Supose that $\wp_1, \dots, \wp_j, P_1, \dots, P_l$ are maximal among the associated primes of $(g_1, \dots, g_k)$ such that f_{k+1} is in $\wp_1, \dots, \wp_j$ and not in $P_1, \dots, P_l$. Also from the *I-depth* consideration (for example, see [K], Chapter VI, 3.1, pp. 183), it follows that I is not contained in these associate primes. Now let

$$\lambda \in I^r \cap P_1 \cdots \cap P_l \backslash \wp_1 \cup \wp_2 \cup \cdots \cup \wp_j.$$

Now let $g_{k+1} = f_{k+1} + \lambda$. Then $g_1, \dots, g_k, g_{k+1}$ is a regular sequence. So, by induction we can find a regular sequence $g_1, \dots, g_n$ such that $g_i - f_i \in I^r$ for $i = 1$ to n.

So, we have $I = (g_1, \dots, g_n) + I^2$ and $J = (g_1, \dots, g_{n-1}) + I^r$. Since J is locally defined by $g_1, \dots, g_{n-1}, g_n^r$, we have J is a locally complete intersection ideal.

For positive integers k, define $J_k = (g_1, \dots, g_{n-1}) + I^k$. Consider the exact sequences

$$0 \to J_k/J_{k+1} \to A/J_{k+1} \to A/J_k \to 0.$$

Also note that the map $g_n^k \colon A/I \to J_k/J_{k+1}$ defines an isomorphism. So, it follows that $[A/J] = [A/J_r] = r[A/I]$.

To prove the converse, we consider the *J-depth* and follow the same proof. □

LEMMA 2.4. *Let A be commutative Noetherian ring and*

$$J_0 = (f_1, f_2, \dots, f_n) + J_0^2$$

be a locally complete intersection ideal of height n. For $k = 1$ to n let

$$J_k = (f_1^{r_1}, \dots, f_{k-1}^{r_{k-1}}, f_{k+1}, \dots, f_n) + J_{k-1}^{r_k}.$$

Then J_n is a locally complete intersection ideal and $[A/J_n] = r_1 r_2 \dots r_n [A/J_0]$ in $K_0(A)$.

Proof. The proof follows from the above Lemma 2.3. □

THEOREM 2.1. *Let $k = \mathbb{Z}$ or a field and let*

$$K = K(k) = \frac{k[S, T, U, V]}{(SU + TV - 1)},$$
$$A = A_n(k) = \frac{k[S, T, U, V, X_1, \dots, X_n, Y_1, \dots, Y_n]}{(SU + TV - 1, X_1Y_1 + \dots + X_nY_n - ST)}.$$

The images of 'upper-case-letters elements' *in A will be denoted by the corresponding* 'lower-case-letters'. *Now let $J_0 = (x_1, \dots, x_n, t)$. Let $r_1, \dots, r_n$ be positive integers. For $k = 1$ to n define*

$$J_k = (x_1^{r_1}, \dots, x_{k-1}^{r_{k-1}}, x_{k+1}, \dots, x_n) + J_{k-1}^{r_k}.$$

Write $J = J_n$. Then

$$J = (x_1^{r_1}, x_2^{r_2}, \dots, x_n^{r_n}) + J^2.$$

Assume $r_1 r_2 \dots r_n$, is divisible by $(n-1)!$. Then there is a projective A-module P of rank n such that P maps onto J and

$$[P] - n = -\frac{r_1 r_2 \dots r_n}{(n-1)!}[A/J_0] \quad in\ K_0(A).$$

Proof. It follows that J_s is generated by $(x_1^{r_1}, \dots, x_n^{r_n})$. Since $(x_1^{r_1}, \dots, x_n^{r_n})$ is a unimodular row in A_{st}, by Suslin's theorem ([S] or see [Ma2]), there is an invertible matrix α in $M_n(A_{st})$ with the transpose of $(x_1^{r_1}, \dots, x_n^{r_n})$ as it's first column. We can assume that α is in $M_n(A)$ with $\det(\alpha) = (st)^u$ for some positive integer u. Now let $f_1\colon A_s^n \to J_s$ be the map defined by sending the standard basis to $(x_1^{r_1}, \dots, x_n^{r_n})$. Also let $f_2 : A_t^n \to J_t$ be the map defined by sending the standard basis to $(1, 0, \dots, 0)$. Now by patching f_1 and f_2 by α we can can find a projective A-module P that maps onto J.

Let $\lambda = [A/(x_1, \dots, x_n, t)]$ in $K_0(A)$. In [Ma1], we proved that $K_0(A)$ is freely generated by $[A]$ and λ. So we have, $[P] - n = m\lambda$ for some integer m. We also have seen in [Ma1] that the Chow group $CH^n(A)$ of codimension n-cycles is freely generated by the cycle $\lambda' = [V(J_0)]$ defined by J_0. We have, the nth Chern class

$$C_n([P] - n) = (-1)^n[V(J)] = (-1)^n r_1 \dots r_n \lambda'$$

and by Riemann–Roch theorem

$$C_n([P] - n) = mC_n(\lambda) = m(-1)^{n-1}(n-1)!\lambda'.$$

So, $m = -(r_1 \dots r_n)/(n-1)!$ and the proof is complete. □

3. The Main Results

The following theorem is an extension of one of the main theorems in [Ma1]:

THEOREM 3.1. *Let A be a Noetherian commutative ring. Suppose J_0 is a locally complete intersection ideal of height $n \leqslant \dim(A)$ and I is an ideal with $I + J_0 = A$. Assume that both J_0 and I contain nonzero divisors and*

$$J_0 = (f_1, \dots, f_n) + J_0^2.$$

For positive integers $r_1, \dots, r_n$ define (as in Lemma 2.1)

$$J_1 = (f_2, \dots, f_n) + J_0^{r_1},$$
$$J_2 = (f_1^{r_1}, f_3, \dots, f_n) + J_1^{r_2},$$
$$\dots\dots\dots$$
$$J = J_n = (f_1^{r_1}, \dots, f_{n-1}^{r_{n-1}}) + J_{n-1}^{r_n}.$$

Suppose Q is a projective A-module of rank n and $\varphi\colon Q \to IJ$ is a surjective map. Assume that $(n-1)!$ divides $r_1 r_2 \dots r_n$. Then

(1) *There is a porjective A-module P of rank n that maps onto J and*

$$[P] - n = -\frac{r_1 r_2 \dots r_n}{(n-1)!}[A/J_0] \quad \text{in } K_0(A).$$

(2) *There is a surjective map $\psi\colon Q \oplus A^n \to I \oplus P$.*

(3) *There is a projective A-module Q' of rank n that maps onto I with*

$$[Q'] = [Q] + \frac{r_1 r_2 \dots r_n}{(n-1)!}[A/J_0] \quad \text{in } K_0(A).$$

Proof. The proof is very similar to that of Theorem 3.1 in [Ma1]. We will only give a sketch of the proof.

Using some prime avoidance arguments, we can find a nonzero divisor $s \in A$ and a free basis $e_1, \ldots, e_n$ of Q_s such that, possibly after modifying $f_1, f_2, \ldots, f_r$, we have the following:

(1) $s \in I$ and $su + t = 1$ for some $t \in J_0$ and $u \in A$.
(2) $sJ_0 \subseteq (f_1, \ldots, f_n)$. So, $st = g_1 f_1 + \cdots + g_n f_n$ for some $g_1, \ldots, g_n$ in A.
(3) $\varphi(e_1) = f_1^{r_1}, \ldots, \varphi(r_n) = f_n^{r_n}$.
(4) We also have $s^k Q \subseteq A^r = \oplus_{i=1}^n Ae_i$.
(5) By replacing Q by $s^k Q$ and I by $s^k I$, we have $s^{k+1} \in I$.
(6) Also, there is an inclusion map $i\colon\ Q \to A^r = \oplus_{i=1}^n Ae_i$ such that $Q_s = A_s^n$.

It is fairly simple to achieve the above in case when A is an integral domain. Now let $A_n(\mathbb{Z})$ be as in Theorem 2.1 and consider the natural map $A_n(\mathbb{Z}) \to A$ that sends X_i to f_i, Y_i to g_i and T to t, S to s, U to u and V to 1. By Suslin's theorem (see [Ma2]) there is a matrix $\gamma \in \mathbf{M_n}(A_n)$ with $\det(\gamma) = s^l k^l$ and transpose of $(x_1^{r_1}, \ldots, x_n^{r_n})$ as it's first column.

Let $\alpha \in \mathbf{M_n}(A)$ be the image of γ. We consider α as a map $\alpha\colon\ A^n \to A^n$ and let $\alpha_0 : Q \to A^r$ be the restriction of α.

Now we construct P as in Theorem 2.1, by patching A_s^n and A_t^n, via α_{st}. It follows that

$$[P] - n = -\frac{r_1 \ldots r_n}{(n-1)!}[A/J_0]$$

because it is the image of the corresponding equation in Theorem 2.1.

Now we have the following observations:

(1) There is an exact sequence

$$0 \to K \to P \to J \to 0.$$

And $K/s^l K$ is free of rank $n-1$, for any positive integer l.

(2) There is an exact sequence

$$0 \to L \to P \oplus I \to J + I = A \to 0.$$

(3) There is a surjective map $g\colon\ P \to J$ and $L \approx g^{-1}(IJ)$.
(4) There is a surjective map $Q \oplus A^{n-1} \to L$.
(5) So, there is a surjective map $Q \oplus A^n \to I \oplus P$ and (2) is estblished.
(6) The proof of (3) follows from (2).

This finishes the sketch of the proof of Theorem 3.1. □

The following theorems on decomposition of projective modules are consequences of the above theorem.

THEOREM 3.2. *Let A be a Noetherian commutative ring and let $f_1, f_2, \dots, f_n$ be a regular sequence with $n \leqslant \dim(A)$. Suppose that $r_1, \dots, r_n$ are positive integers so that $(n-1)!$ divides $r_1 r_2 \dots r_n$. Suppose Q is a projective A-module of rank n and there is a surjective map $\varphi : Q \to (f_1^{r_1}, f_2^{r_2}, \dots, f_n^{r_n})$. Then $[Q] = [Q_0] + 1$ in $K_0(A)$ for some projective A-module Q_0 of rank $n-1$. Further, if $n = \dim(A)$ is odd, A is Cohen–Macaulay and Q has trivial determinant then $Q \approx Q_0 \oplus A$.*

Proof. To prove the first part of the theorem, we apply the above theorem (3.1) with $J_0 = (f_1, \dots, f_n)$ and $I = A$. By (3) of Theorem 3.1, there is a projective A-module Q' of rank n that maps onto A and

$$[Q'] = [Q] + \frac{r_1 r_2 \dots r_n}{(n-1)!}[A/J_0].$$

So, $Q' \approx Q_0 \oplus A$ for some projective A-module Q_0 of rank $n-1$. Since $[A/J_0] = 0$, we have $[Q_0] + 1 = [Q]$.

Now, the latter part follows from [RS], Theorem 4.2. □

THEOREM 3.3. *Let A be a Noetherian commutative ring. Suppose*

$$J_0 = (f_1, \dots, f_n) + J_0^2$$

is a locally complete intersection ideal of height n and

$$J = J_n = (f_1^{r_1}, \dots, f_n^{r_n}) + J^2$$

is defined as in the above Theorem 3.1. Let Q be a projective A-module of rank n and let $\varphi\colon Q \to J$ be a surjective map. Assume that $r_1 r_1 \dots r_n$ is divisible by $(n-1)!$. Then

$$[Q] = [Q_0 \oplus A] - \frac{r_1 \dots r_n}{(n-1)!}[A/J_0]$$

in $K_0(A)$ for some projective A-module Q_0 of rank $n-1$. In particular, if $[A/J_0] = 0$ then $[Q] = [Q_0 \oplus A]$.

Proof. By Theorem 3.1, with $I = A$ there is a surjective map $\psi\colon Q \oplus A^n \to A \oplus P$ where P is a projective A-module of rank n with

$$[P] - n = -\frac{r_1 \dots r_n}{(n-1)!}[A/J_0].$$

The theorem follows with $Q_0 = \ker(\psi)$. □

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