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Erratum

Erratum to "On the complete intersection conjecture of Murthy", with an example in complete intersections [J. Algebra 458 (2016) 156–170]

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ABSTRACT

This is an erratum to [4] and, as well, to [3]. This article provides examples to indicate inconsistencies, in [3], [4].

1. Introduction

The following is a version of the complete intersection conjecture of M. P. Murthy ([9], [5, pp. 85]).

Conjecture 1.1. *Suppose $A = k[X_1, X_2, \dots, X_n]$ is a polynomial ring over a field k. Then, for any ideal I in A, $\mu(I) = \mu\left(\frac{I}{I^2}\right)$*, where μ denotes the minimal number of generators.

A companion to Conjecture 1.1 is the following open problem ([4, 1.2]).

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Open Problem 1.2. *Suppose $A = R[X]$ is a polynomial ring over a noetherian commutative ring R and I is an ideal of A, containing a monic polynomial. Then,* $\mu(I) = \mu\left(\frac{I}{I^2}\right)$

A solution of this conjecture (1.1) was claimed in [3], when k is an infinite perfect field and $1/2 \in k$, which was retracted recently [2]. The claimed proof of Conjecture 1.1 in [3] was a consequence of a stronger claim that, for integers $r \geq 2$, any set of r-generators of I/I^2 lifts to a set of r-generators of I. A version of this question of liftability of generators of I/I^2 was considered in [7], and a counter example was given [7, Example 2.4], when $n = 2$. The community was first alerted, independently, by Mrinal K. Das and this author, regarding the existence of [7, Example 2.4], contrary to the claims of liftablility [3, Theorem 3.2.8]. In Section 2, we develop a larger class of such examples, for all integers $r \geq 2$.

The methods in [3], while erroneous, were vastly generalized in [4], by using Popescu's Desingularization Theorem ([10,12]) and by the homotopy triviality argument [4, Proposition 4.1]. In particular, a solution to the Problem 1.2 was claimed in [4], when R is a regular ring containing a field k, with $1/2 \in F$. We retract all that. Further, the weaker version of S. Abhyankar's Epimorphism Conjecture [4, 1.5] remains open, contrary to the claims in [4].

The claims in [3,4] were fairly striking and fundamental in nature. For the benefit of the wider community, it is imperative that we provide some comprehensive clarity. We do the same in Section 3. We underscore that, there is no logical error in the methods in [4], barring the use of the claimed results in [3].

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2. The examples

The following two examples were worked out in collaboration with M. P. Murthy.

Example 2.1. Let $n \geq 3$ be any integer, and $A = k[X_1, \ldots, X_n; Y_1, \ldots, Y_n]$ be a polynomial ring over any field k. Let $f = \sum_{i=1}^{n} X_i Y_i - 1 \in A$ and $I = Af$. Write $\overline{A} = \frac{A}{(f)}$. For elements in A (respectively, in I), the images in $\overline{A}$ (respectively, in $\frac{I}{I^2}$) will be denoted by "overline". Then,

$$\overline{X_1 f}, \overline{X_2 f}, \quad \ldots, \quad \overline{X_n f} \quad \text{generates} \quad \frac{I}{I^2}.$$

This set of generators of $\frac{I}{I^2}$ would not lift to a set of generators of I.

Proof. As in [7], we have the commutative diagram

$$\begin{tikzcd}
A \arrow[r, "f", "\sim"'] \arrow[d, two heads] & I \arrow[d, two heads] \\
\frac{A}{(f)} \arrow[r, "\sim", "\overline{f}"'] & \frac{I}{I^2}
\end{tikzcd}$$

Suppose $\overline{X_1 f}, \overline{X_2 f}, \quad \ldots, \quad \overline{X_n f}$ lifts to a set of generators of I. Then, by the diagram above, the unimodular row

$$\left(\overline{X_1}, \overline{X_2}, \quad \ldots, \quad \overline{X_n}\right) \qquad \text{of} \quad \overline{A}$$

lifts to a unimodular row

$$(F_1, F_2, \ldots, F_n) \qquad \text{of} \quad A.$$

Since projective A-modules are free, there is a matrix $\sigma \in GL_n(A)$, whose first row is $(F_1, \ldots, F_n)$. Therefore, $\left(\overline{X_1}, \overline{X_2}, \quad \ldots, \quad \overline{X_n}\right)$ is the first row of the image of σ in $GL_n\left(\overline{A}\right)$. So, the projective $\overline{A}$-module defined by $\left(\overline{X_1}, \overline{X_2}, \quad \ldots, \quad \overline{X_n}\right)$ is free. This is impossible, by the Theorem of N. Mohan Kumar and Madhav V. Nori (see [12, Theorem 17.1]). This completes the proof. □

The following is a variation of Example 2.1.

Example 2.2. Let $A = \mathbb{R}[X_0, X_1, \ldots, X_n]$ be a polynomial ring over the field of real numbers. Let $f = \sum_{i=0}^{n} X_i^2 - 1 \in \mathbb{R}$ and $I = Af$. Assume, $n \neq 0, 1, 3, 7$. Then, $X_0 f, X_1 f, \ldots, X_n f$ induce a set of generators for I/I^2, which would not lift to a set of generators of I.

Proof. Same as the proof of (2.1), while we use the fact that tangent bundles over real n-spheres ($n \neq 0, 1, 3, 7$) are nontrivial (see [12, Theorem 2.3]). □

Remark 2.3. Note $I = Af$ in (2.1) is a principal ideal. So, Examples 2.1, 2.2, do not provide a counter example of the Complete Intersection Conjecture 1.1.

3. Erratum

The following list provides some clarifications regarding the inconsistencies in the literature [3,4], at this time.

1. The claimed results [3, Theorem 3.2.7, 3.2.8] in [3] are false, as stated, because Examples 2.1, 2.2 would be contrary to these statements.
2. It was pointed out to me, by Mrinal K. Das that the proof of [3, Lemma 3.2.3] is not convincing. Clearly, this would be a source of logical gap in the arguments in the proof of [3, Theorem 3.2.7].

3. The proof of [3, Theorem 1.0.6] requires further clarifications, with regard to the agreement of the two definitions of Elementary Orthogonal Groups, given in [1] and [11]. Experts seem noncommittal regarding such agreement.
4. The claimed proof of [3, Theorem 3.2.9] is not valid, since it uses [3, Theorem 3.2.8]. Therefore, the Complete Intersection Conjecture 1.1 is still open and the best result on this conjecture, at this time, remains those in [8] and [6].
5. There is no logical error in [4]. However, since the main results in [4] depends on the validity of the same in [3], they do not have any valid proofs, at this time. In particular,
   (a) The claimed results [4, Theorems 3.8, 3.9, 4.2,] are, in deed, false. The claimed result [4, Theorems 4.3] does not have a valid proof. Therefore, the Open Problem 1.2 remains open and the best results on this problem remains those in [6].
   (b) The claimed proof of the weaker version of Abhyankar's epimorphism conjecture [4, Theorem 4.5, 4.6], is not valid.
6. The results in [4] that are not dependent on results in [3] are valid. In particular,
   (a) The [4, Propositions 4.1], on triviality of homotopy obstructions, for ideals containing a monic polynomial, remains valid.
   (b) Results in [4, Section 5], on the alternate description of the Homotopy Obstruction set $\pi_0(Q_{2n})(A)$, remains valid.

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