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# Foxby-morphism and derived equivalences

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ABSTRACT

For quasi-projective schemes $X$ over affine schemes $Spec(A)$, resolving subcategories $\mathscr{A}$ of $Coh(X)$ were considered. The equivalences $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A})) \xrightarrow[\sim]{\iota} \mathscr{D}_g^k((\mathbb{M}_g^0(\mathscr{A})) \xleftarrow[\sim]{\iota'} \mathscr{D}_g^k(\mathscr{A})$ of derived categories were established, where $\mathbb{M}_g^k(\mathscr{A}) = \{\mathcal{F} \in Coh(X) : \dim_{\mathscr{A}}(\mathcal{F}) < \infty, grade(\mathcal{F}) \geq k\}$ and $\mathscr{D}^k$ denote the corresponding filtration of the derived category.



## 1. Introduction

The main theorem in this article establishes equivalences of certain subcategories of the bounded derived category $\mathcal{D}^b(Coh(X))$ of complexes of coherent sheaves over a quasi-projective scheme $X$ over a noetherian affine scheme $Spec(A)$. These subcategories

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concern the derived categories of resolving subcategories $\mathscr{A}$ of $Coh(X)$. For a definition of such a resolving subcategory we refer to (3.1). The category $\mathscr{V}(X)$ of locally free sheaves on $X$ would be the most familiar example of a resolving subcategory. Literature regarding the resolving subcategories outside the realm affine schemes is scarce. Given such a resolving subcategory $\mathscr{A}$ of $Coh(X)$ and integer $k \geq 0$, the main theorem establishes the following equivalence of categories

$$\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A})) \xrightarrow[\sim]{\iota} \mathscr{D}_g^k((\mathbb{M}_g^0(\mathscr{A})) \xleftarrow[\sim]{\iota'} \mathscr{D}_g^k(\mathscr{A})$$

where $\mathbb{M}_g^k(\mathscr{A})$ denotes the category of sheaves $\mathcal{F}$ with $\dim_{\mathscr{A}}(\mathcal{F}) < \infty$ and $grade(\mathcal{F}) \geq k$, $\mathcal{D}^b(*)$ denotes the bounded derived category and $\mathscr{D}_g^k(*)$ denotes the subcategory of complexes in $\mathcal{D}^b(*)$ with grade of the homologies at least $k$. The statement would be most intuitive, when $\mathscr{A} = \mathscr{V}(X)$ and $X$ is locally Cohen–Macaulay, in which case grade, locally, is the height of the annihilator. If $\mathscr{A} = \mathcal{M}CM(X)$ is the category of maximal Cohen–Macaulay sheaves on $X$, then it follows

$$\mathcal{D}^b(Coh_g^k(X)) \xrightarrow[\sim]{\iota} \mathscr{D}_g^k(Coh(X))) \xleftarrow[\sim]{\iota'} \mathscr{D}_g^k(\mathcal{M}CM(X)) \ .$$

The equivalence $\mathcal{D}^b(Coh_g^k(X)) \xrightarrow[\sim]{\iota} \mathscr{D}_g^k(Coh(X)))$ is a result of Keller [6], where filtration by codimension of support was used, instead of grade.

In the case, when $X = Spec(A)$ is affine and Cohen–Macaulay, Sane and Sanders [11] established these equivalences. Methods in [11] relies on a construction of a map $K_\bullet \longrightarrow M_\bullet$ of complexes of modules from some direct sum of (exact) Koszul complexes $K_\bullet$ to a given complex $M_\bullet$ of modules, so that the nonzero homology of $K_\bullet$ surjects onto the corresponding homology of $M_\bullet$. The construction was originally due to H.-B. Foxby [3] that appeared in a preprint. By now, several other expositions and versions of the same is available in the literature [4,15,10,11]. To meet the goals of this paper, a similar morphism of complexes of coherent sheaves on quasi-projective schemes $X$ was constructed (2.3). Other than that, the proof of the main theorem adapts the inductive arguments in [11], which involves further technicalities and finesse in this non-affine situation. Overall, these methods emanate from the methodologies developed in [7–9] in the context of Witt theories, by constructing cones, with smaller width, of morphisms of complexes. There are multiple applications of these equivalences to Witt theory and $\mathbb{K}$-theory that we address in Section 4. While such applications are routine in some cases, they encompass a wide range of categories (see Section 4.3).

Regarding layout of this article, Section 2 deals with extending the morphism of Foxby [3] to quasi-projective schemes. The main equivalence theorem was considered in Section 3. Some of the consequences were discussed in Section 4. *In much of the arguments, there would be no loss of generality if the complexes are given a translation. That is why, in many statements and proofs, we considered degree zero as the generic reference degree.*

## 2. The Foxby morphism

In this section we extend the chain complex map, originally due to Foxby [3], to quasi-projective schemes. Following is a routine extension of the process of selecting regular sequences in an ideal in a ring. Refer to (3.1) for the definition of grade, used in this section.

**Lemma 2.1.** *Let $X$ be an open subset of $\tilde{X} := Proj(S)$, for some noetherian graded ring $S$ and $\dim X = d$. Let $Y \subseteq X$ be a closed subset of $X$, with $grade(\mathcal{O}_Y, X) \geq k$. Let $V(I) = \overline{Y}$ be the closure of $Y$, where $I$ is the homogeneous ideal of $S$, defining $\overline{Y}$. Then, there is a sequence of homogeneous elements $f_1, \ldots, f_k \in I$ such that $f_{i_1}, \ldots, f_{i_j}$ induce regular $S_{(\wp)}$-sequences $\forall\ \wp \in Y \subseteq X$, and $\forall\ 1 \leq i_1 < i_2 < \cdots < i_j \leq k$.*

**Proof.** We only do the inductive step. Now suppose $t < k$ and there is a sequence $f_1, \ldots, f_t \in I$ that induce regular sequences in $S_{(\wp)}\ \forall\ \wp \in Y$. We let $\mathcal{P}_t = \left\{\wp \in \tilde{X} : \wp \in Ass(f_{i_1}, \ldots, f_{i_s}) \cap X : 1 \leq i_1 < \cdots < f_s \leq t\right\}$. Claim, for $\wp \in \mathcal{P}_t$, $I \nsubseteq \wp$. To see this, suppose $I \subseteq \wp \in Ass(f_{i_1}, \ldots, f_{i_s}) \cap X \subseteq \mathcal{P}_t$. Simplifying notations, assume $I \subseteq \wp \in Ass(f_1, \ldots, f_s) \cap X$. Then, $I_\wp \subseteq \wp S_{(\wp)} \in Ass(f_1, \cdots, f_s)$; which is a contradiction because $grade(I_{(\wp)}) \geq k$. So, we can choose $f_{t+1} \in I \setminus \bigcup \mathcal{P}_t$. The proof is complete. $\square$

We recall the following definition for the purpose of setting up notations.

**Construction 2.2.** Suppose $S = \oplus S_n$ is a noetherian commutative graded ring with $S_0 = A$. Let $f_1, f_2, \ldots, f_k \in S_\kappa$ be homogeneous with $\deg(f_i) = \kappa$ for all $i$. Then Koszul complex $K_\bullet(f_1, f_2, \ldots, f_k)$ of graded modules is defined, as usual, as a complex of graded modules. By shifting degrees, it is assumed that all maps are of degree zero. Sheafifying, $K_\bullet(f_1, \ldots, f_k)$, we get the Koszul complexes $\mathcal{K}_\bullet(f_1, \ldots, f_k)$, of locally free sheaves, on $Proj(S) = \tilde{X}$. Recall, for $0 \leq s \leq k$, $\mathcal{K}_s(f_1, \ldots, f_k) = \bigoplus_{1 \leq i_1 < \cdots < t_s \leq k} \mathcal{O}_{\tilde{X}}(-s\kappa) e_{i_1} \wedge \cdots \wedge e_{i_s}$. Generally, $\mathcal{K}_\bullet(f_1, f_2, \ldots, f_k)$ is exact only on at points $\wp \in \tilde{X}$ such that $f_1, f_2, \ldots, f_k$ induces a regular sequence in $\mathcal{O}_{x,\wp} = S_{(\wp)}$. We will often, write $K_\bullet := K_\bullet(f_1, \ldots, f_k)$, and $\mathcal{K}_\bullet := \mathcal{K}_\bullet(f_1, \ldots, f_k)$. However, we will be working with Koszul complexes of $f_1^n, \ldots, f_r^n$, of varying exponent $n$ and length $r$.

The following is the extension of Foxby's construction to quasi-projective schemes.

**Theorem 2.3.** *Suppose $X$ is a quasi-projective scheme over a noetherian affine scheme $Spec(A)$, with $\dim X = d$. Let*

$$\mathcal{G}_{k+1} \longrightarrow \mathcal{G}_k \xrightarrow[\partial_k]{} \cdots \longrightarrow \mathcal{G}_r \xrightarrow[\partial_r]{} \mathcal{G}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}_{-1}$$

*be a complex of coherent $\mathcal{O}_X$-modules. Assume $\forall\ i = 0, \ldots, k$ $grade(\mathcal{O}_{Y_i}, X) \geq k$, where $Y_i = Supp(\mathcal{H}_i(\mathcal{G}_\bullet)) \subseteq X$. Then, there is a morphism $\nu_\bullet : \mathcal{E}_\bullet \longrightarrow \mathcal{G}_\bullet$ where*

$$\mathcal{E}_\bullet : \ 0 \longrightarrow \mathcal{E}_k \longrightarrow \cdots \longrightarrow \mathcal{E}_r \xrightarrow[d_r]{} \mathcal{E}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow 0$$

*is in $Ch^b(\mathscr{V})$ such that*

1. $\mathcal{H}_i(\mathcal{E}_\bullet) = 0 \ \forall \ i \neq 0$ *and* $\mathcal{H}_0(\nu) : \mathcal{H}_0(\mathcal{E}_\bullet) \twoheadrightarrow \mathcal{H}_0(\mathcal{G}_\bullet)$ *is surjective.*
2. $\mathcal{E}xt^i\left(\mathcal{H}_0(\mathcal{E}_\bullet), \mathcal{O}_X\right) = 0 \ \forall \ i \neq 0$ *and* $\dim_{\mathscr{A}}\left(\mathcal{H}_0(\mathcal{E}_\bullet)\right) = k$. *In fact,* $\mathcal{E}_\bullet$ *would be a direct sum of twisted Koszul complexes that resolves* $\mathcal{F} := \mathcal{H}_0(\mathcal{E}_\bullet)$.

**Proof.** First $X$ is an open subset of $\tilde{X} = Proj(S)$, where $S = A \oplus S_1 \oplus S_2 \oplus \cdots = A[x_0, x_2, \ldots, x_N]$ is a graded ring and $\deg(x_i) = 1$. Assume that the closure $\overline{X} = \tilde{X}$. It would seem more intuitive (*though avoidable*), if we extend $\mathcal{G}_\bullet$ to a complex (see [2, II §5])

$$\mathcal{F}_\bullet : \ 0 \longrightarrow \mathcal{F}_n \longrightarrow \cdots \longrightarrow \mathcal{F}_r \xrightarrow[\partial_r]{} \mathcal{F}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{F}_0 \longrightarrow \mathcal{F}_{-1}$$

over $\tilde{X}$. Let $Y = \bigcup Supp(\mathcal{H}_i(\mathcal{G}_\bullet))$ and $\tilde{Y} = \bigcup Supp(\mathcal{H}_i(\mathcal{F}_\bullet))$. Then, $\tilde{Y} = V(I)$ for some homogeneous defining ideal $I$ in $S$. By (2.1), there is a sequence $f_1, \ldots, f_k$ such that they induce a regular sequence in $S_{(\wp)}$ for all $\wp \in Y$. Assume $\deg(f_i) = \kappa$ is constant and $f_i$ kills all the homologies $\mathcal{H}_i(\mathcal{F}_\bullet)$. For the rest of this proof, let $k$ be fixed. Write $\varphi_{i,m} = f_i^m$. So, $\deg(\varphi_{i,m}) = m\kappa$. Let $K_{\bullet,m} = K(\varphi_{1,m}, \ldots, \varphi_{k,m})$ be the graded Koszul complex. Then $K_{k,m} = S(-km\kappa)$.

We denote $B_i(\mathcal{F}_\bullet) = image(\partial_{i+1}) \subseteq Z_i(\mathcal{F}_\bullet) = \ker(\partial_i) \subseteq \mathcal{F}_i$ and use similar notations for any complex. No generality is lost, if we replace $\mathcal{G}_\bullet$ by $\mathcal{G}_\bullet(m) := \mathcal{G}_\bullet \otimes \mathcal{O}_{\tilde{X}}(m)$ for any $m \in \mathbb{Z}$. So, we will assume $B_i(\mathcal{F}_\bullet), Z_i(\mathcal{F}_\bullet), \mathcal{F}_i$ are globally generated and (see [2, III.5.2]) $H^1(\tilde{X}, Z_i(\mathcal{F}_\bullet(n))) = 0 \ \forall \ n \geq 0$. Now let $s \in \Gamma(\tilde{X}, Z_0(\mathcal{F}))$ and $\alpha_0 : \mathcal{O}_{\tilde{X}} \longrightarrow Z_0(\mathcal{F})$ sending $1 \mapsto s$. Since $\Gamma(\tilde{X}, Z_0(\mathcal{F}))$ finitely generated, it would be enough to extend $\alpha_0$ as required.

For all $1 \leq r \leq k$, $m \geq r$ and $1 \leq i_1 < \ldots < i_r$, denote $\mathcal{K}_{\bullet,m,i_1\cdots i_r} := K_\bullet(\varphi_{i_1,m}, \ldots, \varphi_{i_r,m})$. By induction, we will prove that

$$\exists \ \text{a map} \ \ \nu : \mathcal{K}_{\bullet,m,i_1\cdots i_r} \longrightarrow \mathcal{F}_\bullet \ \ \text{such that} \ \ \nu_0 = \alpha_0. \tag{1}$$

First, we prove it for $r = 1$. Since $f_i\mathcal{H}_i(\mathcal{F}_\bullet) = 0$, the $image(f_i^m\alpha_0) \subseteq B_0(\mathcal{F}_\bullet)$. It would suffice to prove it for $i = 1$. Consider the diagram

$$\begin{array}{ccccc} & & \mathcal{O}_{\tilde{X}}(-m\kappa) & \xrightarrow{f_1^m} & \mathcal{O}_{\tilde{X}} \\ & & \downarrow & & \downarrow{\scriptstyle \alpha_0} \\ \mathcal{F}_1 & \underset{d_1}{\twoheadrightarrow} & B_0(\mathcal{F}_\bullet) & \hookrightarrow & \mathcal{F}_0 \end{array}$$

The twisted map $f_1^m\alpha_0 \otimes \mathcal{O}_{\tilde{X}}(m\kappa) : \mathcal{O}_{\tilde{X}} \longrightarrow B_0(\mathcal{F}_\bullet)(m\kappa))$ is given by the global section $\epsilon := f_1^m\alpha_0 \otimes \mathcal{O}_{\tilde{X}}(m\kappa)(1) \in \Gamma(\tilde{X}, B_0(\mathcal{F}_\bullet)(m\kappa)) = H^0(\tilde{X}, B_0(\mathcal{F}_\bullet)(m\kappa))$. Consider the exact sequence $0 \longrightarrow Z_0(\mathcal{F}_\bullet)(m\kappa)) \longrightarrow \mathcal{F}_0(m\kappa) \longrightarrow B_0(\mathcal{F}_\bullet)(m\kappa)) \longrightarrow 0$, and the cohomology exact sequence:

$$H^0(\tilde{X}, \mathcal{F}_0(m\kappa)) \longrightarrow H^0(\tilde{X}, B_0(\mathcal{F}_\bullet)(m\kappa)) \longrightarrow H^1(\tilde{X}, Z_0(\mathcal{F}_\bullet)(m\kappa)) = 0.$$

So, $\epsilon \in H^0(\tilde{X}, B_0(\mathcal{F}_\bullet)(m\kappa))$ lifts to a section $\epsilon' \in H^0(\tilde{X}, \mathcal{F}_0(m\kappa))$. Define $\alpha'_1 : \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{F}(m\kappa)$ by sending $1 \mapsto \epsilon'$. Then, $\alpha_1 = \alpha'_1 \otimes \mathcal{O}_{\tilde{X}}(-m\kappa)$ fits in the above commutative diagram. This completes the proof for $r = 1$.

To do the induction, we assume that the statement (1) has been proved for $r = k - 1$. For all $j = 1, \ldots, k$ denote $\mathcal{K}_{\bullet,m,\hat{j}} := \mathcal{K}_\bullet(\varphi_{1,m}, \ldots, \varphi_{j-1,m}, \varphi_{j+1,m}, \ldots, \varphi_{n,m})$. By induction hypothesis, for $m \geq k - 1$, $j = 1, \ldots, k$, we have a map $K_{\bullet,m,\hat{j}} \longrightarrow \mathcal{F}_\bullet$ as in (1). Combining all these we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathcal{O}_{\tilde{X}}(-mk\kappa) & \longrightarrow & \bigoplus_{j=1}^{k} \mathcal{O}_{\tilde{X}}(-m(k-1)\kappa) & \longrightarrow & \cdots & \longrightarrow & \mathcal{O}_{\tilde{X}} & \longrightarrow 0 \\
& \downarrow \beta & & \downarrow \alpha_{k-1} & & & & \downarrow \alpha_0 & \\
\mathcal{F}_k \twoheadrightarrow & B_{k-1}(\mathcal{F}_\bullet) & \longrightarrow & \mathcal{F}_{k-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{F}_0 & \longrightarrow \mathcal{F}_{-1}
\end{array} \tag{2}$$

Here the top line is the Koszul complex $\mathcal{K}(\varphi_{m,1}, \ldots, \varphi_{m,k})$. A priory, $\beta$ maps into $Z_{k-1}(\mathcal{F}_\bullet)$. However, $\beta$ maps into $B_{k-1}(\mathcal{F}_\bullet)$, which needs a proof. This is proved locally and follows from the proof in the affine case (*e.g.* [11] *has an explicit proof*). The map $\beta(mk\kappa) : \mathcal{O}_{\tilde{X}} \longrightarrow B_{k-1}(\mathcal{F}_\bullet)(mk\kappa)$ is given by a global section $\epsilon \in H^0\left(\tilde{X}, B_{k-1}(\mathcal{F}_\bullet)(mk\kappa)\right)$. Consider the exact sequence

$$0 \longrightarrow Z_k(\mathcal{F}_\bullet)(mk\kappa) \longrightarrow \mathcal{F}_k(mk\kappa) \longrightarrow B_{k-1}(\mathcal{F}_\bullet)(mk\kappa) \longrightarrow 0$$

and its homology exact sequence

$$H^0\left(\tilde{X}, \mathcal{F}_k(mk\kappa)\right) \longrightarrow H^0\left(\tilde{X}, B_{k-1}(\mathcal{F}_\bullet)(mk\kappa)\right) \longrightarrow H^1\left(\tilde{X}, B_{k-1}(\mathcal{F}_\bullet)(mk\kappa)\right) = 0.$$

So, $\epsilon$ lifts to a global section $\epsilon' \in H^0\left(\tilde{X}, \mathcal{F}_k(mk\kappa)\right)$. Define $\alpha' : \mathcal{O}_{\tilde{X}} \longrightarrow \mathcal{F}_k(mk\kappa)$ by sending $1 \mapsto \epsilon'$. Let $\alpha_k = \alpha'(-mk\kappa)$, which fits in the commutative diagram (2). While the Koszul complex on the top line of (2) need not be exact on $\tilde{X}$, its restriction on $X$ is. The proof is complete. □

**Remark 2.4.** In the proof of (2.3), one had the choices of the sequence $f_1, \ldots, f_k$, as required above. We will exploit this flexibility later.

## 3. The equivalence theorem

Before we state and prove the equivalence theorem, we set up some notations.

**Notations 3.1.** Throughout this article, $X$ will denote a quasi-projective scheme over a noetherian affine scheme $Spec(A)$ and $d := \dim X$. We introduce further notations.

1. Throughout, $\mathscr{A}$ will denote a resolving subcategory of $Coh(X)$ and $\mathscr{V}(X)$ will denote the category of all locally free sheaves on $X$. Recall [8] a subcategory $\mathscr{A}$ of $Coh(X)$ is called a resolving subcategory if it is closed under direct summand, extensions and kernel of epimorphisms. However, we will further assume that all resolving subcategories $\mathscr{A}$, we consider, contain $\mathscr{V}(X)$. Denote $\mathbb{M}(\mathscr{A}) = \{\mathcal{F} \in Coh(X) : \dim_{\mathscr{A}}(\mathcal{F}) < \infty\}$.
2. In this article, we consider filtration of $Coh(X)$ and $\mathbb{M}(\mathscr{A})$ by grade. Recall, for $\mathcal{F} \in Coh(X)$, $grade(\mathcal{F}) := \min\{t : \mathcal{E}xt^t(\mathcal{F}, \mathcal{O}_X) \neq 0\}$. For integers $k \geq 0$, denote $Coh^k_g(X) := Coh^k(X) := \{\mathcal{F} \in Coh(X) : grade(\mathcal{F}, \mathcal{O}_X) \geq k\}$ and $\mathbb{M}^k_g(\mathscr{A}) := \mathbb{M}^k(\mathscr{A}) := \{\mathcal{F} \in \mathbb{M}(\mathscr{A}) : grade(\mathcal{F}, \mathcal{O}_X) \geq k\}$. So, we have a filtration $\mathbb{M}(\mathscr{A}) = \mathbb{M}^0_g(\mathscr{A}) \supseteq \mathbb{M}^1_g(\mathscr{A}) \supseteq \cdots \supseteq \mathbb{M}^d_g(\mathscr{A}) \supseteq 0$. Throughout, we will strictly be using this filtration by garde. Note that $\mathbb{M}^k_g(\mathscr{A})$ is a Serre subcategory (2 out of 3) of $\mathbb{M}(\mathscr{A})$. When $X$ is locally Cohen–Macaulay, this filtration is same as the filtration by co-dimension of the support. When $\mathscr{A} = \mathscr{V}(X)$, $\mathbb{M}(\mathscr{A}) = \mathbb{M}^0(\mathscr{A})$ is the category of coherent sheaves in $Coh(X)$, with finite locally free dimension. The bounded derived category of an exact category $\mathscr{E}$ will be denoted by $\mathcal{D}^b(\mathscr{E})$. Similarly, $Ch^b(\mathscr{E})$ will denote the category of chain complexes. Most importantly, for $\mathscr{E} = \mathscr{A}, \mathbb{M}(\mathscr{A})$, let $\mathscr{D}^k_g(\mathscr{E})$ denote the derived subcategory of $\mathcal{D}^b(\mathscr{E})$, of complexes $\mathcal{E}_\bullet$ such that all the homologies $\mathcal{H}_i(\mathcal{E}_\bullet) \in Coh^k_g(X)$. (*Note the difference between two fonts* $\mathcal{D}, \mathscr{D}$.)

The following is the statement of the main equivalence theorem.

**Theorem 3.2.** *Suppose $X$ is a quasi-projective scheme over a noetherian affine scheme $Spec(A)$ and $\dim X = d$. Consider the commutative diagram of natural functors*

$$\begin{array}{ccccc} \mathcal{D}^b(\mathbb{M}^k_g(\mathscr{A})) & \xrightarrow{\iota} & \mathscr{D}^k_g((\mathbb{M}(\mathscr{A})) & \xleftarrow{\iota'} & \mathscr{D}^k_g(\mathscr{A}) \\ & & \downarrow & & \downarrow \\ & & \mathcal{D}^b(\mathbb{M}(\mathscr{A})) & \xleftarrow[\iota'']{} & \mathcal{D}^b(\mathscr{A}) \end{array} \tag{3}$$

*Then, the functors $\iota$, $\iota'$, $\iota''$ are natural equivalences.*

**Proof.** The $\iota''$ is the case of $k = 0$, of $\iota'$. Also $\iota'$ is an equivalence, where the inverse functor is obtained by going through the double complexes. So, we are left with proving the $\iota$ is an equivalence. This will be done by the following propositions. □

The following version of [8, Lemma 5.3] would be needed for the proofs below.

**Lemma 3.3.** *Let $\mathscr{C}$ be an abelian category. Let $\mathcal{F}_\bullet, \mathcal{G}_\bullet$ be two objects in $\mathcal{D}^b(\mathscr{C})$. Assume (1) $\mathcal{H}_r(\mathcal{F}_\bullet) = 0 \ \forall r \le n_0 - 1$, (2) $\mathcal{H}_r(\mathcal{G}_\bullet) = 0 \ \forall r \ge n_0$. Then, $Mor_{\mathcal{D}^b(\mathscr{C})}(\mathcal{F}_\bullet, \mathcal{G}_\bullet) = 0$. Further, if $\mathscr{V}$ is a resolving subcategory of $\mathscr{C}$, then same holds for $\mathscr{V}$ and $\mathbb{M}(\mathscr{V})$.*

**Proof.** Now, we prove the first part. Let $\eta_\bullet : \mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$ be a morphism We can assume $n_0 = 0$ and $\eta_\bullet$ is a map of complexes (denominator free). Further, by replacing by a quasi-isomorphic complex, we assume $\mathcal{F}_i = 0 \ \forall \ i \le -1$. Define the subcomplex $\mathcal{G}'_\bullet \hookrightarrow \mathcal{G}_\bullet$ by setting $\mathcal{G}'_i = \mathcal{G}_i \ \forall \ i \ge 1$, $\mathcal{G}'_0 = \ker(\mathcal{G}_0 \longrightarrow \mathcal{G}_{-1})$ and $\mathcal{G}'_i = 0 \ \forall \ i \le -1$. Since $\eta_\bullet$ factors through $\mathcal{G}'_\bullet$, by replacing $\mathcal{G}_\bullet$ by $\mathcal{G}'_\bullet$, we can assume $\mathcal{G}_i = 0 \ \forall \ i \le -1$ and $\mathcal{G}_\bullet$ is exact. Hence, $\mathcal{G}_\bullet \cong 0$ in $\mathcal{D}^b(\mathscr{C})$ and $\eta_\bullet = 0$. Note, $\mathscr{V} \longrightarrow \mathcal{M}(\mathscr{V})$ is an equivalence of categories and the same proof works in $\mathcal{D}^b(\mathbb{M}(\mathscr{V}))$. The proof is complete. □

Throughout the rest of this article, given an object $\mathcal{F} \in \mathbb{M}^k(\mathscr{A})$ we will use the same notation $\mathcal{F}$ to denote the corresponding complex in $\mathcal{D}^b(\mathbb{M}^k(\mathscr{A}))$, with single nonzero term $\mathcal{F}$ at degree zero. It would be clear from the context whether $\mathcal{F}$ denotes an object or the complex.

**Proposition 3.4.** *The functor $\iota$, in diagram (3), is essentially surjective and full.*

**Proof.** For a complex $\mathcal{F}_\bullet \in Ch^b(Coh(X))$, we say that $width(\mathcal{F}_\bullet) \le r$, if $\mathcal{H}_i(\mathcal{F}_\bullet) = 0$ unless $m \ge i \ge n - r$ for some integer $m, n$. By induction on $r$, we prove

1. Given $\mathcal{F}_\bullet \in \mathscr{D}^k_g(\mathbb{M}(\mathcal{A}))$, with $width(\mathcal{F}_\bullet) \le r$, $\mathcal{F}_\bullet \cong \iota(\tilde{\mathcal{F}}_\bullet)$ for some $\tilde{\mathcal{F}}_\bullet \in \mathcal{D}^b(\mathbb{M}^k_g(\mathscr{A}))$.
2. Given $\mathcal{F}_\bullet, \mathcal{G}_\bullet \in \mathcal{D}^b(\mathbb{M}^k_g(\mathscr{A}))$, with $width(\mathcal{F}_\bullet \oplus \mathcal{G}_\bullet) \le r$ the map $Mor_{\mathcal{D}^b(\mathbb{M}^k_g(\mathscr{A}))}(\mathcal{F}_\bullet, \mathcal{G}_\bullet) \longrightarrow Mor_{\mathscr{D}^k_g(\mathbb{M}(\mathscr{A}))}(\mathcal{F}_\bullet, \mathcal{G}_\bullet)$ is surjective.

Let $r = 0$ and $\mathcal{F}_\bullet$ be as in (1). We can assume that $\mathcal{H}_i(\mathcal{F}_\bullet) = 0$ for all $i \ne 0$. We can further assume that $\mathcal{F}_i = 0 \ \forall \ i \le -1$. It follows, $\mathcal{H}_0(\mathcal{F}_\bullet) \in \mathcal{D}^b(\mathbb{M}^k_g(\mathscr{A}))$, as a complex concentrated at degree zero and $\iota(\mathcal{H}_0(\mathcal{F}_\bullet)) \cong \mathcal{F}_\bullet$.

Similarly, suppose $\mathcal{F}_\bullet, \mathcal{G}_\bullet \in \mathcal{D}^b(\mathbb{M}^k_g(\mathscr{A}))$, $width(\mathcal{F}_\bullet \oplus \mathcal{G}_\bullet) = 0$ and $f : \mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$ is a morphism in $\mathscr{D}^k_g(\mathbb{M}(\mathscr{A}))$. We have, $f = g_\bullet t_\bullet^{-1} : \ \mathcal{F}_\bullet \xleftarrow{t_\bullet} W_\bullet \xrightarrow{g_\bullet} \mathcal{G}_\bullet$ , where $t_\bullet$ is a quasi-isomorphism. It follows that all three can be considered as resolution of their homologies. Consider these homologies as complexes, concentrated at degree zero. Hence $\iota(\mathcal{H}_0(g)\mathcal{H}_0(t)^{-1}) = g_\bullet t_\bullet^{-1} = f_\bullet$. This completes the proof for $r = 0$.

Now suppose $r > 0$ and $\mathcal{F}_\bullet \in \mathscr{D}^k_g(\mathbb{M}(\mathscr{A}))$ as in hypothesis (1). We can assume $\mathcal{F}_0 = 0 \ \forall \ i \le -1$ and $\mathcal{H}_0(\mathcal{F}_\bullet) \ne 0$. By (2.3), there is a complex $\mathcal{E}_\bullet$ and a morphism $\nu : \mathcal{E}_\bullet \longrightarrow \mathcal{F}_\bullet$ such that

1. $\mathcal{E}_\bullet \in Ch^b(\mathscr{V}(X)) \subseteq Ch^b(\mathbb{M}(\mathscr{A}))$ and $\mathcal{E}_i = 0$ unless $k \ge i \ge 0$.
2. $\mathcal{H}_i(\mathcal{E}_\bullet) = 0$ for all $i \ne 0$.

3. $\mathcal{H}_0(\mathcal{E}_\bullet) \in \mathbb{M}_g^k(\mathscr{V}(X))$; in fact $\dim_{\mathscr{A}}(\mathcal{H}_0(\mathcal{E}_\bullet)) = co\dim(\mathcal{H}_0(\mathcal{E}_\bullet)) = k$.
4. $\mathcal{H}_0(\nu) : \mathcal{H}_0(\mathcal{E}_\bullet) \twoheadrightarrow \mathcal{H}_0(\mathcal{F}_\bullet)$ is surjective.

Consider $\mathcal{H}_0(\mathcal{E}_\bullet)$ as complex in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A})$, concentrated at degree zero. Then, $\iota(\mathcal{H}_0(\mathcal{E}_\bullet)) = \mathcal{E}_\bullet$. Embed $\nu$ in an exact triangle in $\mathscr{D}_g^k(\mathbb{M}(\mathscr{A}))$:

$$T^{-1}\Delta_\bullet \xrightarrow{\nu_0} \mathcal{E}_\bullet \xrightarrow{\nu} \mathcal{F}_\bullet \xrightarrow{\nu_2} \Delta_\bullet \ .$$

The corresponding homology exact sequence yields, $\forall\ i \le 0\ \mathcal{H}_i(\Delta_\bullet) = 0$, $\forall\ i \ge 2$ $\mathcal{H}_i(\mathcal{F}_\bullet) \cong \mathcal{H}_i(\Delta_\bullet)$ and

$$0 \longrightarrow \mathcal{H}_1(\mathcal{F}_\bullet) \longrightarrow \mathcal{H}_1(\Delta_\bullet) \longrightarrow \mathcal{H}_0(\mathcal{E}_\bullet) \longrightarrow \mathcal{H}_0(\mathcal{F}_\bullet) \longrightarrow 0$$

is exact. Therefore, $width(\mathcal{H}_i(\Delta_\bullet)) < r$. By induction, there is a complex $\tilde{\Delta}_\bullet \in \mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$ such that $\iota(\tilde{\Delta}_\bullet) = \Delta_\bullet$. Consider $\mathcal{H}_0(\mathcal{E}_\bullet)$ as a complex, concentrated at degree zero. It follows $width(\mathcal{H}_0(\mathcal{E}_\bullet \oplus T^{-1}(\tilde{\Delta}_\bullet)) < r$. Using the induction hypothesis (2), there is a morphism $\eta_0 : T^{-1}\tilde{\Delta}_\bullet \longrightarrow \mathcal{H}_0(\mathcal{E}_\bullet)$ in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$ such that $\iota(\eta_0) = \nu_0$. Now, embed $\tilde{\nu}_0$ in an exact triangle in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$: $T^{-1}\tilde{\Delta}_\bullet \xrightarrow{\eta_0} \mathcal{H}_0(\mathcal{E}_\bullet) \xrightarrow{\eta} \mathcal{U}_\bullet \xrightarrow{\eta_2} \tilde{\Delta}_\bullet$. Now apply $\iota$ to this triangle and complete the diagram:

$$\begin{array}{ccccccc}
T^{-1}\Delta_\bullet & \xrightarrow{\nu_0} & \mathcal{E}_\bullet & \xrightarrow{\nu} & \mathcal{F}_\bullet & \xrightarrow{\nu_2} & \Delta_\bullet \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr\ \epsilon & & \downarrow\wr \\
T^{-1}\iota(\tilde{\Delta}_\bullet) & \xrightarrow[\iota(\eta_0)]{} & \iota(\mathcal{H}_0(\mathcal{E}_\bullet)) & \xrightarrow[\iota(\eta)]{} & \iota(\mathcal{U}_\bullet) & \xrightarrow[\iota(\eta_2)]{} & \iota(\tilde{\Delta}_\bullet)
\end{array}$$

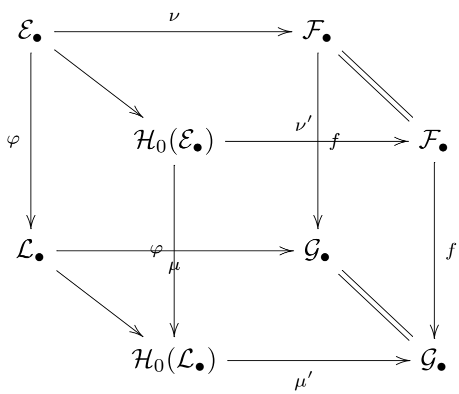
The isomorphism $\epsilon$ is obtained by properties of triangulated categories. This completes the proof of (1).

To complete the inductive step of the proof of (2), suppose $f : \mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$ be a morphism in $\mathcal{D}^k(\mathbb{M}_g^k(\mathscr{A}))$, where $\mathcal{F}_\bullet, \mathcal{G}_\bullet \in \mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$ and $width(\mathcal{F}_\bullet \oplus \mathcal{G}_\bullet) = r$. Assume $\mathcal{F}_i = \mathcal{G}_i = 0$ for all $i \le -1$ and either $\mathcal{H}_0(\mathcal{F}_\bullet) \neq 0$ or $\mathcal{H}_0(\mathcal{G}_\bullet) \neq 0$. In either case, by (2.3), there are complexes $\mathcal{E}_\bullet, \mathcal{L}_\bullet$ and morphisms $\nu : \mathcal{E}_\bullet \longrightarrow \mathcal{F}_\bullet$, $\mu : \mathcal{L}_\bullet \longrightarrow \mathcal{G}_\bullet$ such that

1. $\mathcal{E}_\bullet, \mathcal{L}_\bullet \in Ch^b(\mathscr{V}(X)) \subseteq Ch^b(\mathbb{M}(\mathscr{A}))$ and $\mathcal{E}_i = \mathcal{L}_i = 0$ unless $k \ge i \ge 0$.
2. $\mathcal{H}_i(\mathcal{E}_\bullet) = \mathcal{H}_i(\mathcal{L}_\bullet) = 0$ for all $i \neq 0$.
3. $\mathcal{H}_0(\mathcal{E}_\bullet), \mathcal{H}_0(\mathcal{L}_\bullet) \in \mathbb{M}_g^k(\mathscr{V}(X))$; in fact $\dim_{\mathscr{A}}(\mathcal{H}_0(\mathcal{E}_\bullet)) = grade(\mathcal{H}_0(\mathcal{E}_\bullet)) = k$, $\dim_{\mathscr{A}}(\mathcal{H}_0(\mathcal{L}_\bullet)) = grade(\mathcal{H}_0(\mathcal{L}_\bullet)) = k$.
4. $\mathcal{H}_0(\nu) : \mathcal{H}_0(\mathcal{E}_\bullet) \twoheadrightarrow \mathcal{H}_0(\mathcal{F}_\bullet)$ and $\mathcal{H}_0(\mu) : \mathcal{H}_0(\mathcal{L}_\bullet) \twoheadrightarrow \mathcal{H}_0(\mathcal{G}_\bullet)$ are surjective.
5. Further, in the proof of (2.3), we choose the respective sequences in $Ann(\mathcal{F}_0) \cap Ann(\mathcal{G}_0)$ (*more precisely, in their extensions in the closure* $\tilde{X}$). This choice, will give the following commutative diagrams:

$$\begin{CD} \mathcal{E}_\bullet @>{\nu}>> \mathcal{F}_\bullet \\ @VVV @| \\ \mathcal{H}_0(\mathcal{E}_\bullet) @>{\nu'}>> \mathcal{F}_\bullet \end{CD}\quad ,\qquad \begin{CD} \mathcal{L}_\bullet @>{\mu}>> \mathcal{G}_\bullet \\ @VVV @| \\ \mathcal{H}_0(\mathcal{L}_\bullet) @>{\mu'}>> \mathcal{G}_\bullet \end{CD}\qquad \text{all maps are in } \; Ch^b(Coh(X)).$$

6. By replacing $\mathcal{L}_\bullet$ by $\mathcal{E}_\bullet \oplus \mathcal{L}_\bullet$, we can further assume that the diagram




commutes. (4)

Embed $\mu', \nu'$ in exact triangles ans follows and obtain the morphism of exact triangles:

$$\begin{CD} T^{-1}\Delta_\bullet @>{\nu_0}>> \mathcal{H}_0(\mathcal{E}_\bullet) @>{\nu'}>> \mathcal{F}_\bullet @>{\nu_1}>> \Delta_\bullet \\ @V{T^{-1}\eta}VV @V{\varphi}VV @VV{f}V @VV{\eta}V \\ T^{-1}\Gamma_\bullet @>>{\mu_0}> \mathcal{H}_0(\mathcal{L}_\bullet) @>>{\mu'}> \mathcal{G}_\bullet @>>{\mu_1}> \Gamma_\bullet \end{CD} \tag{5}$$

where the two exact triangles are in $Ch^b(\mathbb{M}^k(\mathscr{A}))$ and the vertical morphisms are in $\mathscr{D}^k_g(\mathbb{M}(\mathscr{A}))$. The induction hypotheses applies to $\eta, \varphi$. So, there are morphisms $\tilde{\eta} : \Delta_\bullet \longrightarrow \Gamma_\bullet$, $\tilde{\varphi} : \mathcal{H}_0(\mathcal{E}_\bullet) \longrightarrow \mathcal{H}_0(\mathcal{L}_\bullet)$ such that $\iota(\tilde{\eta}) = \eta$, $\iota(\tilde{\varphi}) = \varphi$. This gives the following commutative (as clarified below) diagram:

$$\begin{CD} T^{-1}\Delta_\bullet @>{\nu_0}>> \mathcal{H}_0(\mathcal{E}_\bullet) @>{\nu'}>> \mathcal{F}_\bullet @>{\nu_1}>> \Delta_\bullet \\ @V{T^{-1}\tilde{\eta}}VV @V{\tilde{\varphi}}VV @VV{g}V @VV{\tilde{\eta}}V \\ T^{-1}\Gamma_\bullet @>>{\mu_0}> \mathcal{H}_0(\mathcal{L}_\bullet) @>>{\mu'}> \mathcal{G}_\bullet @>>{\mu_1}> \Gamma_\bullet \end{CD} \qquad \text{in } \; \mathcal{D}^b(\mathbb{M}^k_g(\mathscr{A})). \tag{6}$$

It is clear that image of the left square in $\mathscr{D}^k_g(\mathbb{M}(\mathscr{A}))$ commutes. Claim that the left square commutes. This, a special case of faithfulness, is proved below as Lemma 3.5. Therefore, there is a morphism $g$ in $\mathcal{D}^b(\mathbb{M}^k_g(\mathscr{A}))$, obtained by properties of triangulated categories. Apply $\iota$ to (6) and compare with (5). With $h = f - \iota(g)$ we obtain the commutative diagram

$$\begin{array}{ccccccc} T^{-1}\Delta_\bullet & \xrightarrow{\nu_0} & \mathcal{H}_0(\mathcal{E}_\bullet) & \xrightarrow{\nu'} & \mathcal{F}_\bullet & \xrightarrow{\nu_1} & \Delta_\bullet \\ \downarrow 0 & & \downarrow 0 & \swarrow \zeta & \downarrow h & \swarrow \epsilon & \downarrow 0 \\ T^{-1}\Gamma_\bullet & \xrightarrow[\mu_0]{} & \mathcal{H}_0(\mathcal{L}_\bullet) & \xrightarrow[\mu']{} & \mathcal{G}_\bullet & \xrightarrow[\mu_1]{} & \Gamma_\bullet \end{array} \quad \text{in } \mathscr{D}^k(\mathbb{M}^0(\mathscr{A})),$$

where $\zeta$ and $\epsilon$ are given by weak kernel and weak cokernel properties. Consider the case $\mathcal{H}_0(\mathcal{F}_\bullet) = 0$. Then, by Lemma 3.3, $Mor(\mathcal{F}_\bullet, \mathcal{H}_0(\mathcal{L}_\bullet)) = 0$. Therefore, $\zeta = 0$. Hence $h = 0$ and $f = \iota(g)$. So, it is established, whenever $\mathcal{H}_0(\mathcal{F}_\bullet) = 0$ and $width(\mathcal{F}_\bullet, \mathcal{G}_\bullet) \le r$, then $Mor_{\mathcal{D}^b(\mathbb{M}^k(\mathscr{A}))}(\mathcal{F}_\bullet, \mathcal{G}_\bullet) \twoheadrightarrow Mor_{D^k(\mathbb{M}(\mathscr{A}))}(\mathcal{F}_\bullet, \mathcal{G}_\bullet)$ is surjective. This fact will be used in the next step.

Now, assume $\mathcal{H}_0(\mathcal{F}_\bullet) \neq 0$ and $\mathcal{H}_0(\mathcal{G}_\bullet) \neq 0$. $width(\Delta_\bullet \oplus \mathcal{G}_\bullet) = r$ and $\mathcal{H}_0(\Delta_\bullet) = 0$. By the above remark, $Mor_{\mathcal{D}^b(\mathbb{M}^k(\mathscr{A}))}(\Delta_\bullet, \mathcal{G}_\bullet) \twoheadrightarrow Mor_{D^k(\mathbb{M}(\mathscr{A}))}(\Delta_\bullet, \mathcal{G}_\bullet)$ is surjective. Therefore, $\iota(\tilde{\epsilon}) = \epsilon$ for some $\tilde{\epsilon} \in Mor_{\mathcal{D}^b(\mathbb{M}^k(\mathscr{A}))}(\Delta_\bullet, \mathcal{G}_\bullet)$. So, $h = \tilde{\epsilon}\nu_1$ is in $Mor_{\mathcal{D}^b(\mathbb{M}^k(\mathscr{A}))}(\Delta_\bullet, \mathcal{G}_\bullet)$. Hence, so is $f = \iota(g) + h$. The proof is complete, because we assumed that either $\mathcal{H}_0(\mathcal{F}_\bullet) \neq 0$ or $\mathcal{H}_0(\mathcal{G}_\bullet) \neq 0$. □

The following special case of faithfulness property of $\iota$ was used in the above proof.

**Lemma 3.5.** *Let $X$ be a quasi-projective scheme, as in (3.2). Let $f : \mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$ be a morphism in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$ such that (a) $\mathcal{H}_i(\mathcal{F}_\bullet) = 0\ \forall\, i \le n_0 - 1$, (b) $\mathcal{H}_i(\mathcal{G}_\bullet) = 0\ \forall\, i \neq n_0$, and (c) the image $\iota(f_\bullet) = 0$ in $\mathscr{D}_g^k(\mathbb{M}(\mathscr{A}))$. Then, $f_\bullet = 0$ in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$.*

**Proof.** Without loss of generality, we can assume $n_0 = 0$ and $\mathcal{G}_\bullet = \mathcal{G}$ is a single term complex concentrated at degree zero and also $\mathcal{F}_0 = 0\ \forall\, i \le -1$. We can write $f_\bullet = g_\bullet t_\bullet^{-1}$ where $t$ is a quasi-isomorphism and $g_\bullet$ is a chain complex map. By replacing $f_\bullet$ by $g_\bullet$, we assume $f_\bullet$ is a chain complex map. Note, with $p : \mathcal{F}_0 \longrightarrow \mathcal{H}_0(\mathcal{F}_\bullet)$, $f_0 = \mathcal{H}_0(f_\bullet)p$. Now $\iota(f_\bullet) = 0$ implies $\mathcal{H}_0(f_\bullet) = 0 : \mathcal{H}_0(\mathcal{F}_\bullet) \longrightarrow \mathcal{G}$. So, $f_0 = 0$ and hence $f_\bullet = 0$. The proof is complete. □

To complete the proof of Theorem 3.2, we need to prove the faithfulness of $\iota$, as follows. In the following, we will use that fact that, in a triangulated category, a morphism $f = 0$ if and only if its cone splits.

**Proposition 3.6.** *The functor $\iota$ is faithful.*

**Proof.** Suppose $f_\bullet : \mathcal{F}_\bullet \longrightarrow \mathcal{G}_\bullet$ be a morphism in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$ such that $\iota(f_\bullet) = 0$. We need to prove $f_\bullet = 0$. Without loss of any generality, we can assume that $f_\bullet$ is in $Ch^b(\mathbb{M}_g^k(\mathscr{A}))$. Embed $f$ is an exact triangle $T^{-1}\Delta_\bullet \xrightarrow{g} \mathcal{F}_\bullet \xrightarrow{f} \mathcal{G}_\bullet \xrightarrow{h} \Delta_\bullet$ in

$\mathcal{D}^b((\mathbb{M}_g^k(\mathscr{A}))$. The triangle maps to $T^{-1}\Delta_\bullet \xrightarrow{g} \mathcal{F}_\bullet \xrightarrow{0} \mathcal{G}_\bullet \xrightarrow{h} \Delta_\bullet$ in $\mathscr{D}_g^k(\mathbb{M}(\mathscr{A}))$. Therefore, there is a split $\eta : \mathcal{F}_\bullet \longrightarrow T^{-1}\Delta_\bullet$, in $\mathscr{D}_g^k(\mathbb{M}(\mathscr{A}))$, of $g$. So, $g\eta = 1_{\mathcal{F}_\bullet}$ in $\mathscr{D}_g^k(\mathbb{M}(\mathscr{A}))$. By (3.4), we can assume that $\eta$ is in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$. Now, embed $g\eta$ in a triangle: $\mathcal{F}_\bullet \xrightarrow{g\eta} \mathcal{F}_\bullet \longrightarrow \Gamma_\bullet \longrightarrow T\mathcal{F}_\bullet$ in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$. This triangle maps to $\mathcal{F}_\bullet = \mathcal{F}_\bullet \longrightarrow \Gamma_\bullet \longrightarrow T\mathcal{F}_\bullet$ in $\mathscr{D}^k(\mathbb{M}(\mathscr{A}))$. Therefore, $\Gamma_\bullet = 0$ in $\mathscr{D}^k(\mathbb{M}(\mathscr{A}))$. That means, $\Gamma_\bullet$ is exact. Hence, $\Gamma_\bullet = 0$ in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$. So, $g\eta$ is an isomorphism in $\mathcal{D}^b(\mathbb{M}^k(\mathscr{A}))$. Therefore $g(\eta(g\eta)^{-1}) = 1$ and hence $\eta(g\eta)^{-1}$ is a split of $g$ in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$. Therefore, $f = 0$ in $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$. The proof is complete. □

**Completing the proof of Theorem 3.2.** Follows directly from Propositions 3.4, 3.6. □

The following is a corollary to the method of proofs above.

**Corollary 3.7.** *Suppose $X$ is a quasi-projective scheme over a noetherian affine scheme $Spec(A)$ and $\dim X = d$. Then, the restrictions of the functors $\iota, \iota'$ in the diagram (3) induce equivalences of categories:*

$$\mathcal{D}^b_{\mathbb{M}(\mathscr{A})}(\mathbb{M}_g^k(\mathscr{A})) \longrightarrow \mathscr{D}^k_{g,\mathbb{M}(\mathscr{A})}((\mathbb{M}(\mathscr{A})) \longleftarrow \mathscr{D}^k_{g,\mathbb{M}(\mathscr{A})}(\mathscr{A})$$

*where the subscript $\mathbb{M}(\mathscr{A})$ indicate the full subcategories of the respective derived categories with homologies in $\mathbb{M}(\mathscr{A})$.*

## 4. Some consequences

The main interest in this study would be, for a noetherian scheme $X$, the category $Coh(X)$ of coherent $\mathcal{O}_X$-modules and the subcategory $\mathscr{V}(X)$ of locally free sheaves, which is a resolving subcategory, if all coherent $\mathcal{O}_X$-modules are quotient of a locally free sheave. To avoid technicalities, we assume, for the rest of the article that $X$ is locally Cohen–Macaulay scheme, with $\dim X = d$ and coherent $\mathcal{O}_X$-modules are quotient of a locally free sheave.

### *4.1. Witt theory*

To address the question of Witt theory, duality needs to be considered. For a noetherian scheme $X$, consider the resolving subcategory $\mathscr{A} := \mathscr{V}(X)$. The following remarks are in order, where the notation $\mathscr{A} := \mathscr{V}(X)$ is used for the purpose of subsequent analogies.

1. The natural duality on $\mathcal{D}^b(\mathscr{A})$ induced by $\mathcal{H}om(-, \mathcal{O}_X)$, will be denoted by $^*$.

2. Then, $\mathbb{M}(\mathscr{A}) = \mathbb{M}_g^0(\mathscr{A})$ represents the category of all coherent sheaves with finite locally free dimension, which does not have a natural duality, nor does $\mathbb{M}_g^k(\mathscr{A})$. The equivalence $\iota_0 := (\iota')^{-1}\iota : \mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A})) \xrightarrow{\sim} \mathscr{D}_g^k(\mathscr{A})$ is the resolution functor.
3. One can pull any (translated) duality in $\mathscr{D}_g^k(\mathscr{A})$, forcibly, via $\iota_0$. In particular, with $\# := T^{k*}$, on $\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$ the most natural duality would be $\mathcal{F}_\bullet^\vee := \iota_0^{-1}(\iota_0(\mathcal{F}_\bullet)^\#)$, for complexes $\mathcal{F}_\bullet \in \mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))$.
4. In fact, for a coherent sheaf $\mathcal{F} \in \mathbb{M}_g^k(\mathscr{A})$, when considered as a complex, $\mathcal{F}^\vee$ would be given by a complex, whose homologies are given by $\mathcal{E}xt^i(\mathcal{F}, \mathcal{O}_X)$. So, in general, $\mathbb{M}_g^k(\mathscr{A})$ is not closed under this induced duality.

Before the statement of Corollary 4.1, the readers are referred to [8,7] for definition of the Witt groups, $W(\mathcal{D}^b_{\mathbb{M}(\mathscr{A})}(\mathbb{M}^d(\mathscr{A})))$, $W^d(\mathscr{D}^d_{g\mathbb{M}(\mathscr{A})}(\mathscr{A}))$, of subcategories of triangulated categories with duality. In the light of the above, the first part of the following is a tautology.

**Corollary 4.1.** *Suppose $X$ is a quasi-projective scheme over a noetherian affine scheme Spec$(A)$, with $\dim X = d$ and $\mathscr{A} = \mathscr{V}(X)$. Then there are isomorphism of Witt groups: $W(\mathcal{D}^b(\mathbb{M}_g^k(\mathscr{A}))) \xrightarrow{\sim} W^k(\mathscr{D}_g^k(\mathscr{A}))$, $W(\mathcal{D}^b_{\mathbb{M}(\mathscr{A})}(\mathbb{M}_g^k(\mathscr{A}))) \xrightarrow{\sim} W^k(\mathscr{D}^k_{g,\mathbb{M}(\mathscr{A})}(\mathscr{A}))$. Now consider the case $k = d := \dim X$. In deed, $\mathbb{M}_g^d(\mathscr{A})$ represent the category of coherent sheaves with grade$(\mathcal{F}) = \dim_{\mathscr{A}}(\mathcal{F}) = d$ and it is closed under the duality. We have the following commutative diagram of isomorphisms of Witt groups.*

$$\begin{array}{ccccc} W((\mathbb{M}_g^d(\mathscr{A})) & \xrightarrow{\sim} & W(\mathcal{D}^b_{\mathbb{M}(\mathscr{A})}(\mathbb{M}_g^d(\mathscr{A}))) & \xrightarrow{\sim} & W(\mathcal{D}^b(\mathbb{M}_g^d(\mathscr{A}))) \\ & & \downarrow \wr & & \downarrow \wr \\ & & W^d(\mathscr{D}^d_{g\mathbb{M}(\mathscr{A})}(\mathscr{A})) & \xrightarrow[\sim]{} & W^d(\mathscr{D}_g^d(\mathscr{A})) \end{array} \tag{7}$$

**Proof.** Since the first statement is obvious (3.2, 3.7), we prove the latter statement. The composition of the two homomorphism in the top line is an isomorphism by the theorem of Balmer [1] and the first one (and hence the second) isomorphism follows from [7, §A]. The two vertical lines are isomorphism from the first part (3.2, 3.7). Hence the homomorphism in the second line is also an isomorphism. The proof is complete. □

The subcategory of objects of $\mathbb{M}_g^k(\mathscr{A})$, closed under the duality $^\vee$ is as follows.

**Lemma 4.2.** *Suppose $\mathcal{F} \in \mathbb{M}_g^k(\mathscr{A})$. Then, $\mathcal{F}^\vee \in \mathbb{M}_g^k(\mathscr{A}) \Longleftrightarrow$ grade$(\mathcal{F}) = \dim_{\mathscr{A}}(\mathcal{F}) = k$. Therefore, $\{\mathcal{F} \in \mathbb{M}_g^k(\mathscr{A}) : \mathcal{F}^\vee \in \mathbb{M}^k(\mathscr{A})\}$ is an exact category with duality $\mathcal{F} \mapsto \mathcal{E}xt^k(\mathcal{F}, \mathcal{O}_X)$. We denote this subcategory by $CMFPD(k)$.*

**Proof.** It is clear *grade*$(\mathcal{F}) = \dim_{\mathscr{A}}(\mathcal{F}) = k \Longrightarrow \mathcal{F}^\vee \in \mathbb{M}^k(\mathscr{A})$. Let $\mathcal{F}^\vee \in \mathbb{M}^k(\mathscr{A})$. By downward induction, we will prove that $\mathcal{F}$ has a $\mathscr{V}$-resolution of length $k$. The argument will be same, if we assume $0 \longrightarrow \mathcal{E}_{k+1} \xrightarrow{d_{k+1}} \mathcal{E}_k \longrightarrow \cdots \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0$

is a $\mathscr{V}$-resolution of $\mathcal{F}$. The dual sequence $0 \longrightarrow \mathcal{E}_0^* \longrightarrow \cdots \longrightarrow \mathcal{E}_k^* \xrightarrow{d_{k+1}^*} \mathcal{E}_{k+1}^* \longrightarrow 0$ is exact at all degrees $i \neq k$ (*strictly speaking, at degree zero*). By local checking, one can see that $co\ker(d_{k+1}) \in \mathscr{V}$. Replacing $\mathcal{E}_k$, by $co\ker(d_{k+1}) \in \mathscr{V}$, we get a resolution of length $k$, of $\mathcal{F}$. The proof is complete. □

Next, we consider Example 4.8, which we put as a proposition.

**Proposition 4.3.** *Suppose $X$ is a Gorenstein scheme, with $\dim X = d$. Consider the resolving category $\mathscr{A} := \mathcal{MCM}(X)$, as in Example 4.8. In the case, the usual filtration by codimension of support agrees with that by grade. So, we will drop the subscript $g$.*

1. *In this case, $\mathbb{M}(\mathscr{A}) = Coh(X)$ and $\mathbb{M}^d(\mathscr{A})$ is the subcategory with support in co-dimension $d$* (all locally finite length sheaves, provided all closed points have same codimension).
2. *Also, an injective resolution $\mathscr{I}_\bullet$ of $\mathcal{O}_X$ is a dualizing complex in, the bounded derived category of quasi-coherent sheaves on $X$. Further, the duality induced by $\mathscr{I}_\bullet$ and the one via $\iota_0$ agrees* (quasi-isomorphic).
3. *However, with $I := \mathscr{I}_{-d}$, $M \longrightarrow Hom(M, I)$ induces the duality on $\mathbb{M}^d(\mathscr{A})$.*
4. *The diagram (7) reduces to the diagram of isomorphisms:*

$$W(Coh^d(X))) \xrightarrow{\sim} W(\mathcal{D}^b(Coh^d(X)))) \xrightarrow{\sim} W^d(\mathscr{D}_g^d(\mathcal{MCM}(X))).$$

The first isomorphism is a result of Gille [5].

**Remark 4.4.** Finally, we comment on Example 4.9. Everything stated above for $\mathscr{A} = \mathscr{V}(X)$ remains valid for $\mathscr{A} = \mathscr{G}_\Omega$ and the commutative digram (7) remains intact.

## 4.2. $\mathbb{K}$-theoretic

Given such equivalences of categories, one may consider $K$-theoretic invariances. We refer to [12] definitions and notations on $K$-theory.

**Theorem 4.5.** *Let $X$ be as in Theorem 3.2 and $\mathscr{A}$ be a resolving subcategory of $Coh(X)$, and $\iota$, $\iota'$ be as in the diagram (3). Then $\iota$ and $\iota'$ induce homotopy equivalences $\mathbb{K}(\mathcal{D}^b(\mathbb{M}^k(\mathscr{A}))) \xrightarrow{\sim} \mathbb{K}(\mathscr{D}^k(\mathbb{M}(\mathscr{A}))) \xleftarrow{\sim} \mathbb{K}(\mathscr{D}^k((\mathscr{A})))$ of the $\mathbb{K}$-theory spectra. Consequently, $\mathbb{K}_i(\mathbb{M}^k(\mathscr{A})) \cong \mathbb{K}_i(\mathcal{D}^b(\mathbb{M}^k(\mathscr{A}))) \cong \mathbb{K}_i(\mathcal{D}^b(\mathscr{A}))\ \forall i \in \mathbb{Z}$ are isomorphisms of $\mathbb{K}$-groups.*

**Proof.** Follows from [12, 30.2.30]. □

**Remark 4.6.** We list two comments on the Gersten Complexes.

1. Following [12], the comments in [11] regarding the existence of Gersten spectral sequences, for (negative) $\mathbb{K}$-theory, extends verbatim for quasi-projective schemes, as considered in this article. While a routine applications of localization theorems would lead to such a spectral sequence, Theorem 3.2 is needed to obtain the desired form.
2. Further, following [13], an existence of similar Gersten spectral sequences, for (negative) Grothendieck–Witt $\mathbb{G}W$-theory, can be established (unpublished).

### *4.3. Examples of resolving categories*

There is a wide range of applications of the equivalence Theorem 3.2, whenever there is a resolving subcategory $\mathscr{A}$ of the $Coh(X)$, for any quasi-projective scheme. As was pointed out in [8], in the affine case, a good amount of literature is available (see [14]), regarding examples of such resolving subcategories, which extends routinely to the non-affine situations. Following is a list of the main examples that are of our special interest.

**Example 4.7.** Suppose $X$ is a noetherian scheme with $\dim X = d$. The motivating example of resolving subcategories of $Coh(X)$ would be $\mathscr{V}(X)$.

**Example 4.8.** Suppose $X$ is a noetherian scheme with $\dim X = d$. The subcategory $\mathcal{M}CM(X)$, of maximal Cohen–Macaulay sheaves $\mathcal{F}$ over $X$, is a resolving subcategory.

**Example 4.9.** Suppose $X$ is a noetherian scheme with $\dim X = d$. Corresponding to any semidualizing sheaf $\Omega$ one can define a resolving subcategory $\mathscr{G}_\Omega$. A coherent sheaf $\Omega \in Coh(X)$ is called a semidualizing sheaf if (1) the morphism $\mathcal{O}_X \xrightarrow{\sim} \mathcal{H}om(\Omega, \Omega)$ is an isomorphism, and (2) $\mathcal{E}xt^i(\Omega, \Omega) = 0 \; \forall \; i \geq 1$. Given such a semidualizing sheaf $\Omega$, for $\mathcal{F} \in Coh(X)$, denote $\mathcal{F}^* := \mathcal{H}om(\mathcal{F}, \Omega)$. Now, let

$$\mathscr{G}_\Omega = \{\mathcal{F} \in Coh(X) : \mathcal{F} \xrightarrow{\sim} \mathcal{F}^{**} \text{ is isomorphism,}$$
$$\mathcal{E}xt^i(\mathcal{F}, \Omega) = \mathcal{E}xt^i(\mathcal{F}^*, \Omega) = 0 \; \forall \; i \geq 1\}$$

Then, $\mathscr{G}_\Omega$ is a resolving subcategory.

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