How Topology shaped, and still shaping, the Obstruction Theory in Algebra

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Abstract

To this date, the obstruction theory for vector bundles in topology shaped the research in projective modules in algebra, almost entirely. The algebra has ever been trying to catch up. Some of us seemed to have taken too long to recognize the importance of topology. More unfortunately, we tried to do it independently.

Rings

- A ring A is a set with an addition $(+)$ and a multiplication. It is a commutative group under addition $+$, and the multiplication is distributive with respect to $+$.
- Any field is a ring. So, $\mathbb{R}, \mathbb{C}$ are rings.
- Let M be a topological space. Let $C(M)$ denote the set of all continuous real valued functions. Then $C(M)$ is a ring. This may be the most inspiring example of a ring.

Modules

- A module M over a ring A is what a vector space would be over a field.
- A free module F over a ring A is an A−module that has a basis. If F is a finitely generated free A−module, then $F \approx A^n$. In this case, define $rank(F) := n$.

Projective Modules

- Suppose A is a commutative ring.
- An $A-$module P is said to be projective, if

$$P \oplus Q = Free$$

for some other $A-$module Q.

Vector bundles

Suppose M is a topological space. A (real) vector bundle on M, is a continuous map $p : \mathcal{E} \to M$ such that

- Each fiber $\mathcal{E}_x = p^{-1}(x)$ has a vector space structure.
- M has an open cover $\{U_i\}$ and homeomorphisms (trivializations) φ_i such that the diagrams

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow[\sim]{\varphi_i} & U_i \times \mathbb{R}^r \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & U_i & \end{array}$$

commute.

- For each $x \in U_i$, the trivialization φ_i induces linear isomorphisms $\mathcal{E}_x \to \mathbb{R}^r$.

Vector bundles

- The rank of $\mathcal{E}$ is defined as $rank(\mathcal{E}) = r$.
- Example: $M \times \mathbb{R}^r \to M$ *is the trivial bundle on* M, *to be denoted by* $\mathcal{R}^r$.
- Example: *The tangent bundle* $\mathcal{T}$ *over a manifold* M, *is a vector bundle.*

The Module of Sections

Let

$$\Gamma(\mathcal{E}) := \{s : M \to \mathcal{E} : ps = Id_M, \quad s \text{ is continuous}\}.$$

This means $s(x) \in \mathcal{E}_x \ \ \forall x \in M$.

1. Elements $s \in \Gamma(\mathcal{E})$ are called sections of $\mathcal{E}$.
2. $\Gamma(\mathcal{E})$ has a natural $C(M)$−module structure.

Noetherian Rings

- The ring $C(M)$ is too big. We work with the ring of algebraic functions only.
- I will often talk about "noetherian commutative rings," because the ring of algebraic functions over a space M are "notherian and commutative".

Never-Vanishing sections

- Let M be a real manifold with $\dim M = d$.
- Let $\mathcal{E}$ be a vector bundle of rank r.
- If $r > d$, then $\mathcal{E}$ has a never-vanishing section. This translates to

$$\Gamma(\mathcal{E}) \approx Q \oplus C(M) \quad as\ C(M) - modules.$$

Splitting

The above inspired the theorem of Serre ([Serre1957]):

- Let A be a noetherian commutative ring with $\dim A = d$.
- Let P be a projective $A-$module of rank r.
- If $r > d$, then P has a free direct summand.

$$\textit{This means} \quad P \approx Q \oplus A.$$

The Correspondence theorem of Swan

Theorem ([Swan 1962])

Suppose M is a (compact connected) Hausdorff topological space. The functor

$$\Gamma : \mathcal{V}(M) \longrightarrow \mathcal{P}(C(M)) \quad \textit{sending} \quad \mathcal{E} \to \Gamma(\mathcal{E})$$

is an equivalence of catagories, where

- *$\mathcal{V}(M)$ denotes the category of vector bundles over M*
- *and $\mathcal{P}(C(M))$ denotes the category of of finitely generated projective $C(M)-$modules.*

The Message

- The message could not have been clearer regarding the connection between vector bundles and Projective Modules.
- Even before this correspondence theorem, smart people saw analogies. Among them would be Serre's Conjecture.

Polynomial rings

- $\mathbb{R}^n$ is contractible. So, vector bundles over $\mathbb{R}^n$ are trivial.
- So, J.-P. Serre conjectured ([Serre1955]) the same for polynomial rings.
- Independently, Quillen and Suslin proved the conjecture:

Polynomial rings

Theorem ([Quillen1976], [Suslin1976])

Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k. Then, finitely generated projective A−modules P are free.

Remarks.

- The conjecture of Serre only drew a simple analogy with the corresponding theorem on Vector bundles.
- Both the proofs of Quillen and Suslin were algebraic, or "at best" geometric.
- However, with a hind sight, I wonder why nobody ever tried to borrow methods from topology?

Search for a newer direction

- The proofs of Quillen and Suslin were rich in techniques and methods that kept many of us busy for more than two decades.
- I went for graduate studies after Serre's conjecture was solved. While the newer techniques were helpful, there was also a sense of stagnation. Question was what to do next?

Chern Classes

- Mohan Kumar and Murthy considered:
  Question: Suppose A is smooth affine algebra over an algebraically closed field k, with $\dim A = d$. Suppose P is a projective $A-$module with $rank(P) = d$.

$$Does \quad C^d(P) = 0 \quad \Longrightarrow \quad P \approx Q \oplus A?$$

  Here $C^d(P)$ denotes the top Chern class of P.
- The question makes sense for any ring A. However, it was always clear that the Chern classes would not be the right obstruction, in general.

Murthy's Theorem

Theorem (Murthy)

Suppose A is an affine algebra over an algebraically closed field k, with $\dim A = d$. *Let P be a projective A−module with* $rank(P) = d$.

$$\textit{Then} \quad C^d(P) = 0 \quad \Longleftrightarrow \quad P \approx Q \oplus A$$

- From the inception to the climax, the whole thing was driven by commutative algebra and algebraic geometry.
- The literature does not give any hint that this project may have had something to do with topology.

Topological Obstructions

- In topology, there is a classical Obstruction theory (see [Steenrod1951]).
- Suppose M is a real smooth manifold with $\dim M = d \geq 2$ and $\mathcal{L}$ is a line bundle over M. Then, there are obstruction groups $\mathcal{H}^n(M, \mathcal{L}) \qquad 0 \leq n \leq d$.
- If $\mathcal{L}$ is trivial these groups turn out to be the singular cohomology groups $H^n(M, \mathbb{Z})$. If $\mathcal{L}$ is non-trivial, they are the cohomology group $H^n(M, \mathcal{G}_{\mathcal{L}})$, with local coefficients in a bundle of groups.

Topological Obstructions

- For a vector bundle $\mathcal{E}$ on M with rank $r \leq d$, there is an invariant
$$w(\mathcal{E}) \in \mathcal{H}^r\left(M, \wedge^r \mathcal{E}\right).$$
- If $\mathcal{E}$ has a never-vanishing section, then $w(\mathcal{E}) = 0$.
- For rank $r = d$, conversely,
$$w(\mathcal{E}) = 0 \Longrightarrow \quad \mathcal{E} = \mathcal{F} \oplus \mathcal{R}.$$

Algebraic Obstructions

In algebra, one had to mimic the existing theory in topology.

- First, around 1989, Madhav V. Nori outlined a program on Obstruction theory in algebra. It was mostly articulated for the case $rank(P) = \dim A$.
- In 2000, Berge and Morel proposed an alternative approach, which is K−theoretic.
- Subsequently, Morel proposed another approach. This is based on a new theory known as "$\mathbb{A}^1$−Homotopy Theory". This seems to be complete mimicry of homotopy theory.

An Overview

Following Nori's outline and iterations, the following emerged.

- Suppose A is a noeth. comm. ring with $\dim A = d \geq 2$ and L is a rank one projective $A-$module. Assume $\mathbb{Q} \subseteq A$. Then, there is an obstruction group $E^d(A, L)$.

- **Theorem ([BhatSri])**

Given a projective $A-$module P of rank d, there is an obstruction class $e(P) \in E\left(A, \wedge^d P\right)$ such that

$$e(P) = 0 \iff P = Q \oplus A.$$

Algebra and topology

Bhatwadekar-Sridharan's theorem was not meant to be a "stand alone" theorem in commutative algebra.

- There had to be a connection to the obstruction thoery in topology. We proceed to discuss the same.
- So, we will consider real affine varieties. That means, those spaces that are defined by vanishing of polynomial functions.

Affine algebras and algebraic varieties

$$\textit{Let} \quad A = \frac{\mathbb{R}[X_1, X_2, \ldots, X_n]}{I} = \mathbb{R}[x_1, x_2, \ldots, x_n],$$

where $I \subseteq \mathbb{R}[X_1, X_2, \ldots, X_n]$ is an ideal of the polynomial functions.

- Let M be the set of points $v \in \mathbb{R}^n$ such that $f(v) = 0$ for all $f \in I$.
- If A is smooth, then $M \subseteq \mathbb{R}^n$ is a smooth maifold. Also $\dim M = \dim A$. (Implicit function theorem.)

Algebra and topology

There are two types of maximal ideals m of A.

- If $A/m \approx \mathbb{C}$ then m is called a complex maximal ideal.
- If $\mathbb{R} \xrightarrow{\sim} A/m$, then m is called a real maximal ideal. In this case, $m = (x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n)$.

$$m \longleftrightarrow (a_1, \ldots, a_n) \in M \quad \textit{is an } 1-1 \textit{ correspondence}$$

between real maximal ideals of A and the points in M.

Structure of the Obstruction Groups

Theorem (Batwadekar-Das-Mandal)

Suppose A is a real smooth affine variety with $\dim A = d$ and M be the corresponding real manifold. Let S be the multiplicative set of all $f \in A$ such that f does not vanish on any real points. Suppose L is any rank one projective $A-$module.

Continued

Then, we have a structure theorem

$$E^d(S^{-1}A, S^{-1}L) \approx \mathbb{Z}^a \times \mathbb{Z}_2^b$$

where

1. a is the number of compact connected component C of M such that the induced bundles $S^{-1} \wedge^d (\Omega_{A/\mathbb{R}})_{|C} \approx S^{-1}L_{|C}$ and
2. b is the number of compact connected component C of M such that $S^{-1} \wedge^d (\Omega_{A/\mathbb{R}})_{|C} \not\approx S^{-1}L_{|C}$.

Algebra and topology

Theorem (Mandal and Sheu): Let $A = \mathbb{R}[x_1, x_2, \ldots, x_n]$ be a smooth algebra over $\mathbb{R}$ and let $M \subseteq \mathbb{R}^n$ be the real manifold, as above. Let $\dim A = \dim M = d \geq 2$ and L be a rank one projective $A-$module and $\mathcal{L}$ be the corresponding line bundle over M.

- Then, there is a canonical homomorphism

$$\epsilon : E(A, L) \to \mathcal{H}^d\left(M, \mathcal{L}^*\right).$$

- For a projective $A-$module P of rank d, we have

$$\epsilon(e(P)) = w(\mathcal{E}^*) \quad \textit{where} \quad \mathcal{E} \quad \textit{is} \quad \textit{the vector bundle}$$

  on M with the module of sections $= P \otimes C(M)$.
- The homomorphism ϵ, factors through an isomorphism

$$E(S^{-1}A, S^{-1}L) \xrightarrow{\sim} \mathcal{H}^d(M, \mathcal{L}^*) \quad \textit{where} \quad S \quad \textit{is}$$

  the set of functions $f \in A$ never vanishing on M.
- **Remark:** In $S^{-1}A$, all the complex maximal ideals of A are killed. So, as sets $Max(S^{-1}A) = M$.

Limitations of Nori's Approach

- The aproach of Nori to Obstruction theory for projective modules was a great success at the top dimension.
- The defintion the Obstruction group $E^d(A, L)$ can be extended to $E^r(A, L)$ routinely. However, there two deficiencies:
  - There is no meaningful way to define the obstruction classes $e(P)$.
  - The groups, so defined, do not fit in some kind of cohomology theory, the way the obstruction groups $\mathcal{H}^r(M, \mathcal{L})$ do.

Paper of Berge and Morel

- In 2000, Berge and Morel proposed an alternative $K-$theoretic approach to the algrbraic obstruction theory.
- Suppose $X = Spec(A)$ is a smooth affine variety over a field k, with $\dim X = d$. Also, let L be a locally free sheaf of rank one.
  - $\forall\ 0 \leq r \leq d$, obstruction groups $E^r(X, L)$ were defined.
  - For projective $A-$modules P of rank r and orientation $\chi : L \xrightarrow{\sim} \wedge^d P$ an obstruction class $e_K(P, \chi) \in E^r(X, L)$ was defined.

Continued

- These groups fit in a cohomology theory.
- Eventually, Fasel proved, if $rank(P) = d$ then

$$e_K(P, \chi) = 0 \quad \Longleftrightarrow \quad P \approx Q \oplus A.$$

Limitations of Berge-Morel Approach

- Vanishing $e_K(P, \chi) = 0$ is not a sufficient condition for splitting free direct summand, if rank $r = rank(P) < d$.
- One needs to assume A is smooth or regular.

Isomorphism classes of vector bundles

Morel proposed an approach which mimics homotopy theory.

- Let M be a smooth real manifold.
- Let $\mathcal{V}_r(M)$ denote the set of all isomorphism classes of vector bundles of rank r, over M.
- (see page 7) There is a natural bijection: $Hom_H(M, B\mathbb{GL}_r(\mathbb{R})) \approx \mathcal{V}_r(M)$.
- Rest is, probably, homotopy theory.

Theory by Analogy

Let $X = Spec(A)$ be a smooth affine variety over a perfect field k. Mimicing the above:

- Let $\mathbb{G}_{r,\infty}(k) = \bigcup \mathbb{G}r_{r,m}(k)$ be the infinite Grassmanian of $r-$planes $\mathbb{A}^r(k)$.
- Let $\Phi_r(X)$ denote the set of all isomorphism classes of projective $A-$modules of rank r.
- (page 230) There is canonical bijection ($r \neq 2$):

$$Hom_{\mathcal{H}(k)}(X, \mathbb{G}_{r,\infty}(k)) \approx Hom_{\mathcal{H}(k)}(X, BGL_r(k)) \approx \Phi_r(X)$$

- Rest would be mimicry of homotopy theory.

What to t expect

- Given a projective $A-$module P of rank r, obstruction for P to split off a free direct summand could be written down. There would be multiple obstructions (thanks to Aravind Asok).

Limitations and Opportunities

- They assume A is smooth.
- There should be a proof of Serre's conjecture using $\mathbb{A}^1-$homotopy theory, and nothing else.
- There should be newer proofs of Eisenbud-Evans Conjectures.

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