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For a commutative ring A and a finitely generated A-module M, we denote

$$\mu(M) := \text{minimal number of generators of M}$$

1 Background and Main Results

We start with the following theorem of Mohan Kumar:

Theorem 1.1. [Mohan Kumar, [Mk]] Suppose $A = R[X]$ is a polynomial ring over a noetherian commutative ring R. Suppose I is an ideal in A that **contains a monic polynomial**.

Assume, $\mu\left(\frac{I}{I^2}\right) \geq \dim\left(\frac{A}{I}\right) + 2$ Then, $\exists$ a surjective map $P \twoheadrightarrow I$

where P is a projective A-module with $rank(P) = \mu\left(\frac{I}{I^2}\right)$.

In particular, suppose $A = k[X_1, \ldots, X_n]$ is a polynomial ring over a field k and I is an ideal in A.

$$\text{Assume} \quad \mu\left(\frac{I}{I^2}\right) \geq \dim\left(\frac{A}{I}\right) + 2 \quad \text{Then,} \quad \mu(I) = \mu\left(\frac{I}{I^2}\right)$$

Subsequently, I proved the following:

Theorem 1.2 (Mandal [M9]). Suppose $A = R[X]$ is a polynomial ring over a noetherian commutative ring R. Suppose I is an ideal in A that **contains a monic polynomial**.

$$\text{Assume} \quad \mu\left(\frac{I}{I^2}\right) \geq \dim(A/I) + 2 \quad \text{Then,} \quad \mu(I) = \mu\left(\frac{I}{I^2}\right)$$

In deed, the following has been a companion to Murthy's original Complete Intersection Conjecture ([M, M8]):

Conjecture 1.3. Suppose $A = R[X]$ is a polynomial ring over a noetherian commutative ring R. Suppose I is an ideal in A that **contains a monic polynomial**. Then,

$$\mu(I) = \mu\left(\frac{I}{I^2}\right)$$

Recall, Murthy's Complete Intersection Conjecture ([M, M8]) is the particular case of the same, when $A = k[X_1, \ldots, X_n]$ is a polynomial ring over a field k.

The following was proved in the recent past:

Theorem 1.4 (Mandal [M5])**.** Let R be a regular ring containing an infinite field k, with $1/2 \in k$. Assume R is essentially smooth over k or k is perfect. Suppose $A = R[X]$ is the polynomial ring and I is an ideal in A that **contains a monic polynomial**.

$$\text{Then,} \qquad \mu(I) = \mu(I/I^2)$$

In fact, any set of n-generators of I/I^2 lifts to a set of generators of I, when $n \geq 2$.

In particular, Murthy's conjecture is settled, in most cases, as follows.

Corollary 1.5 (Mandal)**.** Suppose $A = k[X_1, X_2, \dots, X_n]$ is a polynomial ring over an infinite field k, with $1/2 \in k$. Suppose I is an ideal in A.

$$\text{Then,} \quad \mu(I) = \mu\left(\frac{I}{I^2}\right)$$

Remark: When k is infinite perfect, Fasel proved this result with significant contributions from me.

The weaker version of S. Abhyankar's epi-morphism conjecture [DG] follows from (1.4), as follows. This is significant, because as is indicated in [DG], very limited progress has been made on either version of Abhyankar's epi-morphism conjectures.

Theorem 1.6 (Mandal)**.** Let R be a regular ring over an infinite field k, with $1/2 \in k$. Assume R is essentially smooth over k or k is perfect. Suppose

$$\varphi : R[X_1, X_2, \ldots X_n] \twoheadrightarrow R[Y_1, Y_2, \ldots Y_m] \quad \text{is an epimorphism}$$

of polynomial R-algebras and $I = \ker(\varphi)$. If $n - m \geq \dim R + 1$,

$$\text{Then,} \quad \mu(I) = \mu\left(\frac{I}{I^2}\right)$$

In particular, if R is local, then I is a complete intersection ideal.

The weaker version of S. Abhyankar's epi-morphism conjecture [DG] is settled affirmatively, for infinite fields k, with $1/2 \in k$, as follows.

Corollary 1.7 (Mandal)**.** Suppose k is an infinite field, with $1/2 \in k$. Suppose

$$\varphi : k[X_1, X_2, \ldots X_n] \twoheadrightarrow k[Y_1, Y_2, \ldots Y_m] \quad \text{is an epimorphism}$$

of polynomial k-algebras and $I = \ker(\varphi)$.

$$\text{Then,} \quad \mu(I) = \mu\left(\frac{I}{I^2}\right) = n - m$$

2 Homotopy and Monic

We start with Nori's Homotopy conjecture:

Conjecture 2.1 (M. V. Nori)**.** Suppose $X = \text{Spec}\,(A)$ is a smooth affine scheme over a field k and P is a projective A-module of rank r. Suppose $f_0 : P \twoheadrightarrow I_0$ is a surjective homomorphism, where I_0 is an ideal of A. Now suppose, $I \subseteq A[T]$ is an ideal in the polynomial ring $A[T]$ such that $I(0) = I_0$ and $\varphi : P \otimes A[T] \twoheadrightarrow \frac{I}{I^2}$ is a surjective map, such that φ is compatible with f_0. Then, there is a surjective homomorphism $\psi : P \otimes A[T] \twoheadrightarrow I$ such that $\psi_{|T=0} = f_0$ and ψ lifts φ.

Homotopy is an age old concept, and we give the following definitions:

Definition 2.2. Suppose A is a commutative noetherian ring and P is a projective A-module and I is an ideal of A. A surjective homomorphism $f : \frac{P}{IP} \twoheadrightarrow \frac{I}{I^2}$ would be called a P-local orientation.

$$\text{Let} \quad \mathrm{f_0, f_1} : \frac{\mathrm{P}}{\mathrm{I_iP}} \twoheadrightarrow \frac{\mathrm{I_i}}{\mathrm{I_i^2}} \quad \text{be two P} - \text{local orientations.}$$

We say that f_0 is (strictly) homotopic to f_1, if there is a $P[T]$-local orientation

$$F : \frac{P[T]}{IP[T]} \twoheadrightarrow \frac{I}{I^2} \ni F(0) = f_0 \text{ and } F(1) = f_1$$

Consider the equivalence relation generated by strict homotopy. We say, f_0 is homotopic to f_1, if they are equivalent to each other.

A relaxed version of Nori's Homotopy conjecture 2.1 is the following:

Conjecture 2.3. Use the notations as in (2.3).

$$\text{Suppose} \quad f_0, f_1 : \frac{P}{I_iP} \twoheadrightarrow \frac{I_i}{I_i^2} \quad \text{two P local orientations.}$$

Assume f_0 is (strictly) homotopic to f_1.

$$\text{Suppose} \quad \exists \text{ surjective map } \varphi_0 \ni \quad \begin{array}{ccc} P & \xrightarrow{\varphi_0} & I_0 \\ \downarrow & & \downarrow \\ \frac{P}{I_0P} & \xrightarrow[f_0]{} & \frac{I_0}{I_0^2} \end{array} \quad \text{commutes.}$$

Then, same is true about f_1.

Remark. We are asking whether such lifting property respects homotopy. If $P = A^n$ is free in (2.3), we are talking about the property of lifting of generators of $\frac{I_i}{I_i^2}$ to generators of I_i. When I arrived in Grenoble to visit the author of [F], in May 2015, he invited me to work with him to prove this case when P is free (2.3). I immediately told him that this is only a version of Nori's Homotopy conjecture, which he was not properly conversant with. The following were some of my immediate feedback:

1. Unless A is regular (i. e. unless Bass-Quillen works), existing methods would not apply. So, we focused on such regular rings.

2. Now suppose a noetherian commutative ring and $I \subseteq A[T]$ is an ideal and

$$I = (f_1(T), f_2(T), \ldots, f_n(T)) + I^2.$$

Then

$$\exists S(T) \in I \ \ni \quad \sum_{i=1}^{n} f_i(T)g_i(T) + S(T)(S(T)-1) = 0$$

In the context of variations of Nori's conjecture, the stumbling block had been that we could not pick $S(T) = s \in A$, which I told him.

3. Then, because of my faith in the invisibility of monic polynomials, it did not take too long for me to figure out the following proposition.

Proposition 2.4. Suppose $R = A[X]$ is a polynomial ring over a commutative ring A and I is an ideal that contains a monic polynomial. Suppose $\omega : R^n \twoheadrightarrow I/I^2$ is a surjective homomorphism (*local orientation*). Then, ω is (strictly) homotopic to $A^n \twoheadrightarrow \frac{A}{A}$ given by $(1, 0, \ldots, 0)$.

Proof. Postpone!

3 The Obstruction presheaf

There are skeptics and enthusiasts regarding $\mathbb{A}^1$-homotopy theory. Lately, I probed into it. However, I am convinced that there is a new way to look at things, while I m not competent to say if it really cracks anything. I understood that they try to look at everything as functors or presheafs, which has some advantages. That is why we would restructure the above definition of homotopy. First, we establish some notations that will be useful throughout this article.

Notations 3.1. Throughout, k will denote a field (or ring), with $1/2 \in k$ and A, R will denote commutative noetherian rings. For a commutative ring A and a finitely generated A-module M, the minimal number of generators of M will be denoted by $\mu(M)$.

We denote

$$q_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z^2, \qquad \tilde{q}_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z(Z-1).$$

Denote

$$Q_{2n} = \operatorname{Spec}(\mathscr{A}_{2n}) \text{ where } \mathscr{A}_{2n} = \frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]}{(\tilde{q}_{2n+1})} \tag{1}$$

and

$$Q'_{2n} = \operatorname{Spec}(\mathscr{B}_{2n}) \text{ where } \mathscr{B}_{2n} = \frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]}{(q_{2n+1} - 1)}. \tag{2}$$

There are inverse isomorphisms

$$\alpha : \mathscr{A}_{2n} \xrightarrow{\sim} \mathscr{B}_{2n} \qquad \beta : \mathscr{B}_{2n} \xrightarrow{\sim} \mathscr{A}_{2n}$$

given by

$$\begin{cases} \alpha(x_i) = \frac{x_i}{2} & 1 \leq i \leq n \\ \alpha(y_i) = \frac{y_i}{2} & 1 \leq i \leq n \\ \alpha(z) = \frac{z+1}{2} \end{cases} \qquad \begin{cases} \beta(x_i) = 2x_i & 1 \leq i \leq n \\ \beta(y_i) = 2y_i & 1 \leq i \leq n \\ \beta(z) = 2z - 1 \end{cases} \tag{3}$$

Therefore, $Q_{2n} \cong Q'_{2n}$.

Definition 3.2. The category of schemes over $\operatorname{Spec}(k)$ will be denoted by $\underline{\mathrm{Sch}}_k$. Also, $\underline{\mathrm{Sets}}$ will denote the category of sets.

Given a scheme $Y \in \underline{\mathrm{Sch}}_k$, the association $X \mapsto \mathcal{H}om(X, Y)$ is a presheaf on $\underline{\mathrm{Sch}}_k$. (**Recall, a presheaf is a contravariant functor**.)

This presheaf is often identified with Y, itself. So, in some literature one may write, Y for the presheaf $\mathcal{H}om(-, Y)$ and $Y(X) := \mathcal{H}om(X, Y)$.

With such an approach, for $X = \operatorname{Spec}(A)$, it follows immediately that, $Q_{2n}(A)$ and $Q'_{2n}(A)$ can be identified with the sets, as follows:

$$Q_{2n}(A) = \left\{ (f_1, \ldots, f_n; g_1, \ldots, g_n; s) \in A^{2n+1} : \sum_{i=1}^{n} f_i g_i + s(s-1) = 0 \right\}$$

$$Q'_{2n}(A) = \left\{ (f_1, \ldots, f_n; g_1, \ldots, g_n; s) \in A^{2n+1} : \sum_{i=1}^{n} f_i g_i + s^2 - 1 = 0 \right\}$$

The homotopy pre-sheaves are given by the pushout diagrams in Sets:

$$\begin{array}{ccc} Q_{2n}(A[T]) & \xrightarrow{T=0} & Q_{2n}(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ Q_{2n}(A) & \longrightarrow & \pi_0\left(Q_{2n}\right)(A) \end{array} \quad \text{and} \quad \begin{array}{ccc} Q'_{2n}(A[T]) & \xrightarrow{T=0} & Q'_{2n}(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ Q'_{2n}(A) & \longrightarrow & \pi_0\left(Q'_{2n}\right)(A) \end{array}$$

The isomorphism $Q_{2n} \cong Q'_{2n}$, induces a bijection $\pi_0\left(Q_{2n}\right)(A) \cong \pi_0\left(Q'_{2n}\right)(A)$.

For any ring A and

$$\mathbf{v} = (f_1, \ldots, f_n; g_1, \ldots, g_n; s) \in Q_{2n}(A), \ let\ I(\mathbf{v}) := (f_1, \ldots, f_n, s)A$$

Also, let $\omega_{\mathbf{v}} : A^n \to \frac{I(\mathbf{v})}{I(\mathbf{v})^2}$ denote the surjective homomorphism defined by $e_i \mapsto f_i + I^2$ where $e_1, \ldots, e_n$ is the standard basis of A^n.

Definition 3.3. Suppose A is a commutative ring and I is an ideal in A. For an integer $n \geq 1$, and a local A^n-local orientation, $\omega : A^n \twoheadrightarrow I/I^2$, would be called a local n-orientation of I.

Let $\mathcal{O}(A,n) = \left\{ (I,\omega) : \omega : A^n \twoheadrightarrow \frac{I}{I^2} \quad \text{is a local n} - \text{orientation} \right\}$

For $(I,\omega) \in \mathcal{O}(A,n)$, write

$$\zeta(I,\omega) := [(f_1,\ldots,f_n; g_1,\ldots,g_n,s)] \in \pi_0\left(Q_{2n}(A)\right)$$

where $\sum_{i=1}^n f_i g_i + s(s-1) = 0$ for some $g_1,\ldots,g_n \in A$ and $s \in I$. Note,

$$(f_1,\ldots,f_n; g_1,\ldots,g_n; s) \in Q_{2n}(A).$$

It was established in [F, Theorem 2.0.7], that this association is well defined. We refer to $\zeta(I,\omega)$, as an obstruction class. Therefore, we have a commutative diagram

$$\begin{array}{ccc} Q_{2n}(A) & & \\ {\scriptstyle\eta}\downarrow & \overset{\zeta}{\searrow} & \\ \mathcal{O}(A,n) & \xrightarrow[\zeta]{} & \pi_0\left(Q_{2n}(A)\right) \end{array}$$

and $\eta(\mathbf{v}) = (I(\mathbf{v}), \omega_{\mathbf{v}})$. Note that we use the same notation ζ for two set theoretic maps.

We comment

1. Note that $Q_{2n}(A) \equiv Hom(A, Q_{2n})$ is a presheaf, while $\mathcal{O}(n,A)$ is not. **This why $Q_{2n}(A)$ wins**, and we want to work with it, instead of $\mathcal{O}(n,A)$.

2. We define $\mathbf{u}, \mathbf{v} \in Q_{2n}(A)$ homotopic, if they have same images in $\pi_0(Q_{2n})(A)$.

   Define $\mathbf{u}, \mathbf{v} \in Q_{2n}(A)$ to be strictly homotopic, if

$$\exists\ F(T) \in Q_{2n}(A[T]) \ \ni\ F(0) = \mathbf{u},\ F(1) = \mathbf{v}$$

Proposition 3.4. Suppose $R = A[X]$ is a polynomial ring over a commutative ring A and I is an ideal that contains a monic polynomial. Suppose $\omega : R^n \twoheadrightarrow I/I^2$ is a surjective homomorphism (*local orientation*). Then $\zeta(I, \omega) = [\mathbf{0}] \in \pi_0(Q_{2n})(R)$, where $\mathbf{0} := (0, 0, \ldots, 0, 0, \ldots, 0) \in Q_{2n}(R)$.

Proof. Let $f_1, \ldots, f_n \in I$ be a lift of ω. Then,

$$I = (f_1, f_2, \ldots, f_n) + I^2$$

We can assume that f_1 is a monic polynomial, with even degree. Now, consider the transformation [M9]:

$$\varphi : A[X, T^{\pm 1}] \xrightarrow{\sim} A[X, T^{\pm 1}] \quad \text{by} \quad \begin{cases} \varphi(X) = X - T + T^{-1} \\ \varphi(T) = T \end{cases}$$

There is a commutative diagram

$$\begin{array}{ccc} A[X] & = & A[X] \\ \downarrow & & \uparrow{\scriptstyle T=1} \\ A[X, T^{\pm 1}] & \xrightarrow[\varphi]{} & A[X, T^{\pm 1}] \end{array}$$

Then, $\varphi(f_1) = f_1(X - T + T^{-1})$ is doubly monic in T, meaning that its lowest and the highest degree terms have coefficients 1.

Let $F_1(X,T) = T^{\deg f_1(X)}\varphi(f_1) \in A[X,T]$. Then, $F_1(X,0) = 1$. Also, for $i = 2, \ldots, n$ write $F_i(X,T) = T^{\delta}\varphi(f_i)$, for some integer $\delta \gg 0$, such that $F_i(X,T) \in TA[X,T]$. Therefore, $F_i(X,0) = 0$. Now, write

$$\mathscr{I}' = \varphi(IA[X,T^{\pm 1}]) \quad \text{and} \quad \mathscr{I} := \mathscr{I}' \cap A[X,T].$$

Since $\frac{A[X,T]}{\mathscr{I}} \xrightarrow{\sim} \frac{A[X,T^{\pm 1}]}{\mathscr{I}'}$, it follows

$$\mathscr{I} = (F_1(X,T), \ldots, F_n(X,T)) + \mathscr{I}^2.$$

Therefore, by Nakayama's Lemma, there is a $S(X,T) \in \mathscr{I}$, such that

$$(1 - S(X,T))\mathscr{I} \subseteq (F_1(X,T), F_2(X,T), \ldots, F_n(X,T)).$$

and hence

$$\sum F_i(X,T)G_i(X,T) + S(X,T)(S(X,T) - 1) = 0$$

for some $G_1, \ldots, G_n \in A[X,T]$. Write $\psi(X,T) =$

$(F_1(X,T), F_2(X,T), \ldots, F_n(X,T); G_1(X,T), \ldots, G_n(X,T); S(X,T))$

Then, $\psi(X,T) \in Q_{2n}(A[X,T])$ and $\mathscr{I}_{|T=1} = I$. Further,

$$\psi(X,1) = (f_1, \ldots, f_n; G_1(X,1), \ldots, G_n(X,1); S(X,1))$$

and

$$\psi(X,0) = (1, 0, \ldots, 0; G_1(X,0), \ldots, G_n(X,0), S(X,0)).$$

By [F, 2.0.10], $\psi(X,0) \sim \mathbf{0} \in Q_{2n}(R)$. Hence, $\psi(X,1) \sim \mathbf{0} \in Q_{2n}(R)$. Therefore,

$$\zeta(I,\omega) = [\psi(X,1)] = [\mathbf{0}] \in \pi_0\left(Q_{2n}(R)\right).$$

The proof is complete. ■

Remark 3.5. *In the light of (3.4), our objective would be to prove if* $\mathbf{v} \in Q_{2n}(A)$ *is homotopically trivial, then the corresponding local* n*-orientation*

$$\omega_{I_{\mathbf{v}}} : A^n \twoheadrightarrow \frac{I_{\mathbf{v}}}{I_{\mathbf{v}}^2} \quad \text{lifts to a surjection} \quad \begin{array}{ccc} A^n & \dashrightarrow & I_{\mathbf{v}} \\ & \underset{\omega_{I_{\mathbf{v}}}}{\searrow} & \downarrow \\ & & \frac{I_{\mathbf{v}}}{I_{\mathbf{v}}^2} \end{array}$$

4 Homotopy and the lifting property

4.1 Elementary Orthogonal group and Lifting

Before we proceed, we define the action of $EO\left(A, q_{2n+1}\right)$ on $Q_{2n}(A)$ and give another definition, for the convenience of subsequent discussions.

Definition 4.1. Fix a commutative ring A. As usual, $EO\left(A, q_{2n+1}\right)$ acts on A^{2n+1}, which restricts to an action on $Q'_{2n}(A)$. Using the correspondences

$$\alpha : Q_{2n}(A) \xrightarrow{\sim} Q'_{2n}(A), \quad \beta : Q'_{2n}(A) \xrightarrow{\sim} Q_{2n}(A)$$

define an action on $Q_{2n}(A)$ as follows:

$$\forall\ \mathbf{v} \in Q_{2n}(A), M \in EO\left(A, q_{2n+1}\right) \quad \text{define} \quad \mathbf{v} * M := \beta\left(\alpha(\mathbf{v})M\right)$$

This action is not given by the usual matrix multiplication. Five different classes of the generators of $EO\left(q_{2n+1}\right)(A)$ and their actions on $Q_{2n}(A)$ are given in [F].

Definition 4.2. *Let A be a commutative ring over k. Let $\mathbf{v} \in Q_{2n}(A)$. We write $\mathbf{v} := (a_1, \ldots, a_n; b_1, \ldots, b_n; s)$. For integers, $r \geq 1$ we say that r-lifting property holds for $\mathbf{v}$, if*

$$I(\mathbf{v}) = (a_1 + \mu_1 s^r, \ldots, a_n + \mu_n s^r) \qquad \text{for some } \mu_i \in A.$$

We say the lifting property holds for $\mathbf{v}$, if

$$I(\mathbf{v}) = (a_1 + \mu_1, \ldots, a_n + \mu_n) \qquad \text{for some } \mu_i \in I(\mathbf{v})^2.$$

Before we allude to the key result in [F, Corollary 3.2.6] (see (4.4)), we record the following homotopy lifting theorem, due to this author (unpublished), that was used crucially in the proof.

Theorem 4.3 (Mandal)**.** *Let R be a regular ring containing a field k. Let*
$H(T) := (f_1(T), \ldots, f_n(T), g_1(T), \ldots, g_n(T), s) \in Q_{2n}(R[T])$, with $s \in R$.
Write $a_i = f_i(0), b_i = g_i(0)$. Write $I(T) = (f_1(T), \ldots, f_n(T), s)$. Also assume $I(0) = (a_1, \ldots, a_n)$. Then,

$$I(T) = (F_1, \ldots, F_n) \quad \ni \quad f_i - F_i \in s^2 R[T]$$

Proof. See [F, Lemma 3.1.2], communicated by myself. ∎

Theorem 4.4. *Suppose A is a regular ring containing a field k, with $1/2 \in k$. Let $n \geq 2$ be an integer. Let $\mathbf{v} \in Q_{2n}(A)$ and $M \in EO\,(A, q_{2n+1})$. Then, $\mathbf{v}$ has 2-lifting property if and only if $\mathbf{v} * M$ has the 2-lifting property.*

Proof. We outline the proof in [F]. It would be enough to assume that M is a generator of $EO\,(A, q_{2n+1})$. There would be five cases to deal with, one for each type of generators of $EO\,(A, q_{2n+1})$, listed in [F, pp 3-4]. Only of them is nontrivial, that is of the case of generators of the type 4 (in the list [F, pp 3-4]). This case follows, mainly from Theorem 4.3 (see [F, Lemma 3.1.2]). In deed, I spotted the gap in the proof of (see [F, Lemma 3.1.2]), in the first version of [F] and communicated to the author of [F], what needs to be done to apply Theorem 4.3. ∎

Remark 4.5. *Note, there is no mention of Homotopy in Theorem 4.4. We will show that homotopy relations reduces to the equivalences defined by the action of $EO(q_{2n+1})$.*

5 Homotopy and the action of $EO(q_{2n+1})$

The following is the quadratic analogue of the result of Ton Vorst [T, pp 507].

Theorem 5.1. *Suppose A is a regular ring containing a field k. Then,*

$$\forall\, \sigma(T) \in O(A[T], q_{2n+1}), \;\; \sigma(0) = 1 \Longrightarrow \sigma(T) \in EO(A[T], q_{2n+1}).$$

Proof. In the case when k is perfect, it follows from the theorem of Stavrova ([S, Theorem 1.3]) on REDUCTIVE groups. I reduce it to the perfect field case, using Popescu's theorem.

I remark that an elementary proof would be possible, without using Stavrova's theorem, exactly as the proof of Ton Vorst ([T, pp 507]). Someone needs to work it out.

Theorem 5.2. *Let A be a essentially smooth algebra over an infinite field k, with $1/2 \in k$. Then, for $n \geq 2$, the natural map*

$$\varphi : \frac{Q'_{2n}(A)}{EO(A, q_{2n+1})} \longrightarrow \pi_0\left(Q'_{2n}\right)(A) \quad \text{is a bijection.}$$

Proof. See my paper. According to an expert on quadratic forms, this lemma is standard, which I am not surprised, because of the structure of the proof. ∎

The following summarizes the final results on homotopy and lifting of generators (also see [F, Theorem 3.2.7]).

Theorem 5.3 (Mandal)**.** Suppose A is a regular ring containing an infinite field k, with $1/2 \in k$. Assume A is essentially smooth over k or k is perfect. Let $n \geq 2$ be an integer. Denote $\mathbf{0} := (0, \ldots, 0; 0, \ldots, 0; 0) \in Q_{2n}(A)$ and let $\mathbf{v} \in Q_{2n}(A)$. Then, the following conditions are equivalent:

1. The obstruction $\zeta\left(I(\mathbf{v}, \omega_{\mathbf{v}})\right) = [\mathbf{0}] \in \pi_0\left(Q_{2n}\right)(A)$.

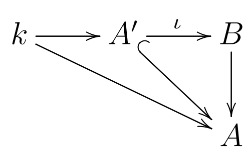
2. $\mathbf{v}$ has 2-lifting property.

3. $\mathbf{v}$ has the lifting property.

4. $\mathbf{v}$ has r-lifting property, $\forall\ r \geq 2$.

Proof. It is clear, (2) $\Longrightarrow$ (3). To prove (3) $\Longrightarrow$ (1), suppose $I(\mathbf{v}) = (a_1 + \mu_1, \ldots, a_n + \mu_n)$, with $\mu_i \in I(\mathbf{v})^2$. Write $\mathbf{v}' = (a_1 + \mu_1, \ldots, a_n + \mu_n; 0, \ldots, 0; 0) \in Q_{2n}(A)$. By [F, 2.0.10], we have $\zeta\left(I(\mathbf{v}, \omega_{\mathbf{v}})\right) = \zeta\left(I(\mathbf{v}', \omega_{\mathbf{v}'})\right) = [v_0] \in \pi_0\left(Q_{2n}\right)$. This establishes, (3) $\Longrightarrow$ (1).

Now we prove (1) $\Longrightarrow$ (2). Assume $\zeta\left(I(\mathbf{v}, \omega_{\mathbf{v}})\right) = [\mathbf{0}]$. In case A is essentially finite over k, it follows from Theorem 5.2 that $\mathbf{0} = \mathbf{v} * M$, for some $M \in EO(A, q_{2n+1})$ and (2) follows from Theorem 4.4. However, when A is regular and contains an infinite perfect field, we have to use Popescu's theorem. By definition, $\zeta\left(I(\mathbf{v}, \omega_{\mathbf{v}})\right) = [\mathbf{0}]$ implies that there is a chain homotopy from $\mathbf{v}$ to $\mathbf{0}$. This data can also be encapsulated in a finitely generated algebra A' over k. As in the proof of (5.1) there is a diagram



of homomorphisms

such that B is smooth over k. The homotopy relations are carried over to B. Therefore, by replacing A by B, we can assume that A is essentially smooth over k. So, Theorem 5.2 applies and (2) follows as in the previous case.

So, it is established that (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3). It is clear that (4) $\Longrightarrow$ (2). Now suppose, one of the first three conditions hold. Fix $r \geq 2$. Notice $I(\mathbf{v}) = (a_1, \ldots, a_n, s^r)A$.

So, replacement of s by s^r leads to the same obstruction class in $\in \pi_0(Q_{2n})(A)$, which is $= [\mathbf{0}] \in \pi_0(Q_{2n})(A)$. Since (1) $\iff$ (2), it follows $I(\mathbf{v})$ has $2r$-lifting property and hence the r-lifting property. The proof is complete. ■

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