GW and K-theory of quasi-projective schemes

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Abstract: We discuss results on K-theory and Grothendieck-Witt (GW) of noetherian quasi-projective schemes X, over affine schemes $\mathrm{Spec}\,(A)$. For integers $k \geq 0$, let $C\mathbb{M}^k(X)$ denote the category of coherent $\mathcal{O}_X$-modules $\mathcal{F}$, with locally free dimension $\dim_{\mathscr{V}(X)}(\mathcal{F}) = k = grade(\mathcal{F})$. We prove that there is a (zig-zag) equivalence $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \to \mathscr{D}^k\left(\mathscr{V}(X)\right)$ of the derived categories. It follows that there is a sequence of zig-zag maps

$$\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}\left(C\mathbb{M}^k(X)\right) \longrightarrow \coprod_{x \in X^{(k)}} \mathbb{K}\left(C\mathbb{M}^k(X_x)\right)$$

of the $\mathbb{K}$-theory spectra that is a homotopy fibration. We also establish similar homotopy fibrations of $\mathbf{GW}$-spectra and $\mathbb{G}W$-bispectra, by application of the same equivalence theorem.

1 Preliminaries and Notations

First, we set up some notations.

Notations 1.1. For an exact category $\mathscr{E}$, $Ch^b(\mathscr{E})$ will denote the category of chain complexes. The bounded derived category of $\mathscr{E}$ will be denoted by $\mathcal{D}^b(\mathscr{E})$.

X will denote a noetherian scheme, with finite dimension $d := \dim X$.

1. For $x \in X$, denote $X_x := \operatorname{Spec}(\mathcal{O}_{X,x})$. Also, $Coh(X)$ will denote the category of coherent $\mathcal{O}_X$-modules and $\mathscr{V}(X)$ will denote the category of all locally free sheaves on X. Denote

$$\mathbb{M}(X) := \mathbb{M}(\mathscr{V}(X)) = \{\mathcal{F} \in Coh(X) : \dim_{\mathscr{V}(X)}(\mathcal{F}) < \infty\}$$

2. We consider filtration of $Coh(X)$ and $\mathbb{M}(X)$ by grade, as opposed to usual filtration by co-dimension of the support.

   Recall, for $\mathcal{F} \in Coh(X)$, $grade(\mathcal{F}) := \min\{t : \mathcal{E}xt^t(\mathcal{F}, \mathcal{O}_X) \neq 0\}$. We remark that, if X is Cohen-Macaulay, then $grade(\mathcal{F}) = co\dim\left(Supp(\mathcal{F})\right)$

   For integers $k \geq 0$, denote

$$\begin{cases} Coh^k(X) := Coh_g^k(X) := \{\mathcal{F} \in Coh(X) : grade(\mathcal{F}) \geq k\} \\ \mathbb{M}^k(X) := \mathbb{M}_g^k(\mathscr{A}) := \{\mathcal{F} \in \mathbb{M}(X) : grade(\mathcal{F}) \geq k\} \\ C\mathbb{M}^k(X) := \{\mathcal{F} \in \mathbb{M}(X) : grade(\mathcal{F}) = k = \dim_{\mathscr{V}(X)}(\mathcal{F})\} \\ \mathscr{C}h^k\left(\mathscr{V}(X)\right) := \left\{\mathcal{F}_\bullet \in Ch^b\left(\mathscr{V}(X)\right) : \forall\ i\ \mathcal{H}_i\left(\mathcal{F}_\bullet\right) \in Coh^k(X)\right\} \\ \mathscr{D}^k\left(\mathscr{V}(X)\right) := \left\{\mathcal{F}_\bullet \in \mathcal{D}^b\left(\mathscr{V}(X)\right) : \forall\ i\ \mathcal{H}_i\left(\mathcal{F}_\bullet\right) \in Coh^k(X)\right\} \\ \mathscr{C}h^k\left(\mathbb{M}(X)\right) := \left\{\mathcal{F}_\bullet \in Ch^b\left(\mathbb{M}(X)\right) : \forall\ i\ \mathcal{H}_i\left(\mathcal{F}_\bullet\right) \in Coh^k(X)\right\} \\ \mathscr{D}^k\left(\mathbb{M}(X)\right) := \left\{\mathcal{F}_\bullet \in \mathcal{D}^b\left(\mathbb{M}(X)\right) : \forall\ i\ \mathcal{H}_i\left(\mathcal{F}_\bullet\right) \in Coh^k(X)\right\} \end{cases}$$

   Here the filtration $\mathbb{M}(X) = \mathbb{M}^0(X) \supseteq \mathbb{M}^1(X) \supseteq \cdots \supseteq \mathbb{M}^d(X) \supseteq 0$, induce filtrations on $\mathcal{D}^b(\mathscr{V}(X))$, $\mathcal{D}^b(\mathbb{M}(X))$, $Ch^b(\mathscr{V}(X))$, $Ch^b(\mathbb{M}(X))$, as above. Clearly, when X is Cohen-Macaulay, this filtration coincides with the filtration by co-dimension of the support.

   For future reference, we remark that $\left(\mathscr{C}h^k\left(\mathscr{V}(X)\right), \mathscr{Q}\right)$, $\left(\mathscr{C}h^k\left(\mathbb{M}(X)\right), \mathscr{Q}\right)$ are **complicial exact categories with weak equivalences** (see [S1] for definition), where weak equivalences being the set $\mathscr{Q}$ of all quasi-isomorphisms.

Example 1.2. If $X = \mathrm{Spec}\,(A)$ is affine and $f_1, \ldots, f_k \in A$ is a regular sequence, then

$$\frac{A}{(f_1, \ldots, f_n)} \in C\mathbb{M}^k(A).$$

Using simple prime avoidance, the following is an useful way to construct objects $\mathcal{F} \in C\mathbb{M}^k(X)$.

Lemma 1.3. Suppose $S = \oplus_{i=0}^{\infty} S_i$, with $S_0 = A$ is a noetherian graded ring over A, $\tilde{X} := Proj(S)$ and $X \subseteq \tilde{X}$ is an open subset (i. e.X *is a quasi projective scheme over* A)

Let $Y \subseteq X$ be a closed subset of X, with $grade\,(\mathcal{O}_Y) \geq k$. Let $V(I) = \overline{Y}$ be the closure of Y, where I is the homogeneous ideal of S, defining $\overline{Y}$. Then, there is a sequence of homogeneous elements $f_1, \ldots, f_k \in I$ such that $f_{i_1}, \ldots, f_{i_j}$ induce regular $S_{(\wp)}$-sequences $\forall\ \wp \in X$, and $\forall\ 1 \leq i_1 < i_2 < \cdots < i_j \leq k$. Write $Z = V(f_1, \ldots, f_k) \cap X$.

1. $\mathcal{F}_n = \mathcal{O}_Z^n \in C\mathbb{M}^k(X)$. In fact, $\oplus_{i=1}^n \mathcal{O}_Z \otimes \mathcal{L}_i \in C\mathbb{M}^k(X)$ for any locally free sheaves $\mathcal{L}_i$ of rank one.

2. Further, if $\mathcal{G} \in Coh^k(X)$, with $Y = Supp\,(\mathcal{G})$ and Z as above, there is a surjective map $\mathcal{F} \twoheadrightarrow \mathcal{G}$ where $\mathcal{F} := \oplus_{i=1}^n \mathcal{O}_Z \otimes \mathcal{O}_x(n_i)$, for some integers n_i. Note that $\mathcal{F} \in C\mathbb{M}^k(X)$.

   In other words, any object $\mathcal{G} \in Coh^k(X)$ is image of an object in $C\mathbb{M}^k(X)$.

Lemma 1.4. *Suppose X is a quasi-projective noetherian scheme over* $\mathrm{Spec}\,(A)$, *with* $\dim X = d$. *Suppose* $\mathcal{F} \in \mathbb{M}^k(X)$. *Then, there is a resolution*

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \xrightarrow{\partial_{n-1}} \cdots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0 \quad \text{with } \mathcal{E}_i \in C\mathbb{M}^k(X). \tag{1}$$

2 The Equivalence Theorems

Lemma 1.3 is to prove following equivalence theorem(s).

Theorem 2.1. Let X be a noetherian quasi-projective scheme over an affine scheme $\mathrm{Spec}\,(A)$ and $k \geq 0$ be a fixed integer. Consider the inclusion functor $C\mathbb{M}^k(X) \hookrightarrow \mathbb{M}^k(X)$. The induced functor

$$\zeta : \mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \longrightarrow \mathcal{D}^b\left(\mathbb{M}^k(X)\right) \quad \text{is an equivalence}$$

of the triangulated categories. Further, consider the inclusion functor $\mathbb{M}^{k+1}(X) \to \mathbb{M}^k(X)$. Then, the induced functor

$$\beta : \mathcal{D}^b\left(\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathcal{D}^b\left(\mathbb{M}^k(X)\right) \quad \text{is fully faithful}$$

Consequently, so are all three below:

$$\begin{array}{ccc} \mathcal{D}^b\left(C\mathbb{M}^{k+1}(X)\right) & & \\ \downarrow & \searrow & \\ \mathcal{D}^b\left(C\mathbb{M}^k(X)\right) & \xrightarrow[\zeta]{\sim} & \mathcal{D}^b\left(\mathbb{M}^k(X)\right) \end{array}$$

Similar arguments and otherwise lead to the following.

Theorem 2.2. Let X be a noetherian quasi-projective scheme as in (2.1) and $k \geq 0$ be a fixed integer. Consider the commutative diagram of functors of derived categories:

$$\begin{array}{ccccccc} \mathcal{D}^b(C\mathbb{M}^{k+1}(X)) & \xrightarrow[\sim]{\zeta} & \mathcal{D}^b(\mathbb{M}^{k+1}(X)) & \xrightarrow[\sim]{\iota} & \mathscr{D}^{k+1}(\mathbb{M}(X)) & \xleftarrow[\sim]{\iota'} & \mathscr{D}^{k+1}(\mathscr{V}(X)) \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \eta \\ \mathcal{D}^b(C\mathbb{M}^{k}(X)) & \xrightarrow[\zeta]{\sim} & \mathcal{D}^b(\mathbb{M}^{k}(X)) & \xrightarrow[\iota]{\sim} & \mathscr{D}^{k}(\mathbb{M}(X)) & \xleftarrow[\iota']{\sim} & \mathscr{D}^{k}(\mathscr{V}(X)) \end{array} \tag{2}$$

Then, all the horizontal functors are equivalences of derived categories and all the vertical functors are fully faithful.

3 Implications

For a noetherian scheme X, denote

$$X^{(k)} := \{Y \in X : co\dim(Y) = k\} \quad \text{and} \quad X_x := \operatorname{Spec}(\mathcal{O}_{X,x}).$$

We will be using the following, to derive the implication of $\mathbb{K}$-theory and $\mathbb{G}W$-theory, which is well known.

Lemma 3.1. Suppose X is a Cohen-Macaulay quasi-projective scheme over an affine scheme $\operatorname{Spec}(A)$ and $k \geq 0$ is an integer. Then, the sequence of derived categories

$$\mathscr{D}^{k+1}(\mathscr{V}(X)) \longrightarrow \mathscr{D}^{k}(\mathscr{V}(X)) \longrightarrow \coprod_{x\in X^{(k)}} \mathcal{D}^k(\mathscr{V}(X_x))$$

is exact up to factor. If X is regular, this sequence is exact.

3.1 K-theory

Our standard reference for K-theory would be the paper of Schlichting [S1], where $\mathbb{K}$-Theory spectra was defined for exact categories $\mathscr{E}$ and for complicial exact categories $\mathscr{E}$ with weak equivalences. For such a category, $\mathbb{K}(\mathscr{E})$ will denote the (negative) $\mathbb{K}$-theory spectra of $\mathscr{E}$ and $\mathbb{K}_i(\mathscr{E})$ will denote the $\mathbb{K}$-groups. Likewise, $\mathbf{K}(\mathscr{E})$ would denote the $\mathbf{K}$-theory space of $\mathscr{E}$.

I have a preference to state the non-connective version of $\mathbb{K}$-Theory, and skip the connective version (K-Theory).

The following is the main application of (2.2) to $\mathbb{K}$-theory.

Theorem 3.2. Suppose X is a noetherian quasi-projective scheme over an affine scheme $\mathrm{Spec}\,(A)$ and $k \geq 0$ is an integer. We consider complicial exact categories with weak equivalences:

$$\begin{cases} \mathscr{C}h^k\,(\mathscr{V}(X)) := \left(\mathscr{C}h^k\,(\mathscr{V}(X))\,, \mathscr{Q}\right) \\ \mathscr{C}h^k\,(\mathbb{M}(X)) := \left(\mathscr{C}h^k\,(\mathbb{M}(X))\,, \mathscr{Q}\right) \end{cases} \qquad \text{likewise.}$$

Consider the diagram of $\mathbb{K}$-theory spectra and maps:

$$\begin{array}{ccccc} \mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) & & \mathbb{K}\left(C\mathbb{M}^{k}(X)\right) & & \coprod_{x\in X^{(k)}} \mathbb{K}\left(C\mathbb{M}^{k}\,(X_x)\right) \\ \wr\downarrow & & \wr\downarrow\Psi & & \downarrow\wr \\ \mathbb{K}\left(\mathscr{C}h^{k+1}\,(\mathbb{M}(X))\right) & \longrightarrow & \mathbb{K}\left(\mathscr{C}h^{k}\,(\mathbb{M}(X))\right) & \longrightarrow & \coprod_{x\in X^{(k)}} \mathbb{K}\left(\mathscr{C}h^{k}\,(\mathbb{M}\,(X_x))\right) \\ \wr\uparrow & & \wr\uparrow\Phi & & \uparrow\wr \\ \mathbb{K}\left(\mathscr{C}h^{k+1}\,(\mathscr{V}(X))\right) & \longrightarrow & \mathbb{K}\left(\mathscr{C}h^{k}\,(\mathscr{V}(X))\right) & \longrightarrow & \coprod_{x\in X^{(k)}} \mathbb{K}\left(\mathscr{C}h^{k}\,(\mathscr{V}\,(X_x))\right) \end{array}$$

Then, the vertical maps are homotopy equivalences of $\mathbb{K}$-theory spectra.

Further, if X is Cohen-Macaulay, then the second line and the third line are homotopy fibrations of $\mathbb{K}$-theory spectra. In particular, the top line is a zig-zag sequence of homotopy fibration of $\mathbb{K}$-theory spectra, of exact categories:

$$\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}\left(C\mathbb{M}^{k}(X)\right) \longrightarrow \coprod_{x\in X^{(k)}} \mathbb{K}\left(C\mathbb{M}^{k}\,(X_x)\right)$$

Proof. Here the middle upward arrow $\Phi : \mathbb{K}\left(\mathscr{C}h^k\,(\mathscr{V}(X), \mathscr{Q})\right) \to \mathbb{K}\left(\mathscr{C}h^k\,(\mathbb{M}(X), \mathscr{Q})\right)$ is induced by the functor $\iota' : \left(\mathscr{C}h^k\left(\mathscr{V}(X), \mathscr{Q}\right)\right) \to \left(\mathscr{C}h^k\left(\mathbb{M}(X), \mathscr{Q}\right)\right)$ of complicial exact categories with weak equivalences. By (2.2), ι' induces an equivalence of the associated triangulated categories. Therefore, by ([S1, 3.2.29]) Φ is a homotopy equivalence. Likewise, other two upward arrows are homotopy equivalences.

The middle downward arrow Ψ is a composition of three maps, as follows:

$$\begin{array}{ccc} \mathbb{K}\left(C\mathbb{M}^k(X)\right) & \xrightarrow{\Psi'} & \mathbb{K}\left(Ch^b\left(C\mathbb{M}^k(X), \mathscr{Q}\right)\right) \\ {\scriptstyle\Psi}\downarrow & & \downarrow{\scriptstyle\zeta'} \\ \mathbb{K}\left(\mathscr{C}h^k\left(\mathbb{M}(X), \mathscr{Q}\right)\right) & \xleftarrow[\iota']{} & \mathbb{K}\left(Ch^b\left(\mathbb{M}^k(X), \mathscr{Q}\right)\right) \end{array}$$

Now, ζ' and ι' are induced by the corresponding functors of complicial exact categories, with weak equivalences. Again, by ([S1, 3.2.29]) in conjunction with Theorem 2.2, ζ' and ι' are homotopy equivalences. Now, Ψ' is a homotopy equivalence by the agreement theorem [S1, 3.2.30]. Hence, so is Ψ.

It remains to show that, when X is Cohen-Macaulay, the third line is a homotopy fibrations of $\mathbb{K}$-theory spectra. To do this, consider sequence of complicial exact categories with weak equivalences (not necessarily exact):

$$\left(\mathscr{C}h^{k+1}\left(\mathscr{V}(X)\right), \mathscr{Q}\right) \longrightarrow \left(\mathscr{C}h^{k}\left(\mathscr{V}(X)\right), \mathscr{Q}\right) \longrightarrow \coprod_{x \in X^{(k)}} \left(\mathscr{C}h^{k}\left(\mathscr{V}(X_x)\right), \mathscr{Q}\right) \qquad (3)$$

By (3.1), the corresponding sequence of the derived categories is exact up to factor. Therefore, by an application of the non-connective version of the Thomason-Waldhausen localization theorem (see [S1, 3.2.27]) the third line in the statement of the theorem is a homotopy fibration of $\mathbb{K}$-theory spectra. The proof is complete. ∎

It is customary to write down the following $\mathbb{K}$-theory exact sequence, which is an immediate consequence of (3.2).

Corollary 3.3. *Let X and k be as in Theorem 3.2. Assume X is Cohen-Macaulay. Then, for any integer n, there is an exact sequence of $\mathbb{K}$-groups,*

$$\cdots \longrightarrow \mathbb{K}_n\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}_n\left(C\mathbb{M}^{k}(X)\right) \longrightarrow \oplus_{x\in X^{(k)}}\mathbb{K}_n\left(\mathbb{M}^{k}(X_x)\right)$$

$$\longrightarrow \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \cdots$$

Proof. Follows from Theorem 3.2. The proof is complete. ∎

Remark 3.4. Let X be as in Theorem 3.2. Assume X is Cohen-Macaulay. The following are some remarks.

The diagram to compute the Gersten complex, reduces to

$$\begin{array}{ccccc}
\oplus_{x\in X^{(k-1)}}\mathbb{K}_{n+1}\left(C\mathbb{M}^{k-1}(X_x)\right) & & & & \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+2}(X)\right) \\
\downarrow & \searrow & & & \downarrow \\
\mathbb{K}_n\left(C\mathbb{M}^{k}(X)\right) & \longrightarrow & \oplus_{x\in X^{(k)}}\mathbb{K}_n\left(C\mathbb{M}^{k}(X_x)\right) & \longrightarrow & \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X)\right) \\
\downarrow & & & \searrow & \downarrow \\
\mathbb{K}_n\left(C\mathbb{M}^{k-1}(X)\right) & & & & \oplus_{x\in X^{(k+1)}}\mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X_x)\right)
\end{array}$$

The dotted diagonal arrows form the Gersten complex.

The spectral sequence given in [B3] takes the following form:

$$E_1^{p,q} = \bigoplus_{x\in X^{(p)}} \mathbb{K}_{-p-q}(C\mathbb{M}^{p}(X_x)) \Longrightarrow \mathbb{K}_{-n}(\mathscr{V}(X)) \qquad \text{along} \quad p+q=n.$$

4 Grothendieck-Witt theory

To do Grothendieck-Witt theory, one needs to incorporate dualities, to what is said above. We refer back to the diagram 2, in Theorem 2.2. The categories

$$\mathbb{M}^k(X),\ \mathcal{D}^b(\mathbb{M}^k(X)),\ \mathbb{M}(X),\ \mathscr{D}^k(\mathbb{M}(X)) \quad \text{have NO natural duality,}$$

$$\text{while} \quad \mathscr{D}^k(\mathscr{V}(X)) \quad \text{has a duality, induced by} \quad \mathcal{H}om\,(-, \mathcal{O}_X)\,.$$

Lemma 4.1. Let X be a noetherian scheme and $k \geq 0$. Then,

$$\mathcal{F} \mapsto \mathcal{E}xt^k\,(\mathcal{F}, \mathcal{O}_X) \quad \text{is a duality on} \quad C\mathbb{M}^k(X).$$

Hence, it induces a duality on the derived category $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right)$.

Proposition 4.2. *Let X be a noetherian quasi-projective scheme, over an affine scheme* $\mathrm{Spec}\,(A)$*, and $k \geq 0$ be an integer. Then, there is a duality preserving equivalence*

$$\mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \longrightarrow T^k\mathscr{D}^k\left((\mathscr{V}(X)\right)$$

of the triangulated categories, where T denotes the shift, the duality on $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right)$ is induced by $\mathcal{E}xt^k\left(-, \mathcal{O}_X\right)$ and that on $T^k\mathscr{D}^k\left((\mathscr{V}(X)\right)$ is $\# := T^k\mathcal{H}om\left(-, \mathcal{O}_X\right)$.

Proof. It is a standard fact that there is a functor $\mathbb{M}(X) \longrightarrow \mathcal{D}^b\left(\mathscr{V}(X)\right)$, by resolution (e.g. see [M1, 3.3]). The restriction to this functor to $C\mathbb{M}^k(X)$ extends to a functor $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \rightarrow \mathscr{D}^k\left((\mathscr{V}(X)\right)$. It turns out that this functor represents the composite functor in (2.2). Hence the functor is an equivalence. Now, routine checking establishes that this functor preserves the duality, as required. The proof is complete. ∎

Again, our main reference to Grothedieck Witt theory is the paper of Schlichting, [S3], the Grothendieck Witt (GW) theory, where the theory was developed for dg categories with weak equivalences and duality. Like K-theory, GW-theory is invariant of equivalences of the associated triangulated categories of the dg-Category, and when 2 is invertible. While Proposition 4.2 provided equivalences at the derived category level, there is no functor form $dgC\mathbb{M}^k(X)$ to $dg\mathscr{V}(X)$. Therefore, we need do a little more work.

Notations 4.3. We establish some notations as follows. Let X denote a noetherian scheme.

1. In analogy to previous notations, for integers $k \geq 0$,

$$\begin{cases} Perf(X) = \text{Category of Perfect complexes of } \mathcal{O}_{\mathrm{X}}\text{modules} \\ \mathcal{D}(Perf(X)) = \text{Derived category of } Perf(X) \\ \mathscr{P}erf^k(X) := \left\{\mathcal{F}_\bullet \in Perf(X) : \forall\, i\ \mathcal{H}_i\left(\mathcal{F}_\bullet\right) \in Coh^k(X)\right\} \\ \mathscr{D}^k Perf(X) := \left\{\mathcal{F}_\bullet \in \mathcal{D}^b(Perf(X)) : \forall\, i\ \mathcal{H}_i\left(\mathcal{F}_\bullet\right) \in Coh^k(X)\right\} \end{cases}$$

To avoid confusion, we will use prefix dg to denote the respective dg categories. So, $dgPerf(X)$ would denote the dg category whose objects are same as that of $Perf(X)$.

2. In fact $Perf(X)$ has a duality. Fix a minimal injective resolution $I_\bullet$ of $\mathcal{O}_X$, as follows:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow I_0 \longrightarrow I_{-1} \longrightarrow I_{-2} \longrightarrow \cdots. \qquad \text{Clearly, } I_\bullet \in Perf(X).$$

For $\mathcal{F}_\bullet \in Perf(X)$, denote $\mathcal{F}^\vee := \mathcal{H}om(\mathcal{F}_\bullet, I_\bullet)$.

The following addresses the duality aspect of $dgPerf(X)$.

Lemma 4.4. *Let X be a noetherian scheme. Let $I_\bullet$ be as in Notation 4.3 (2). Then, the association*

$$\mathcal{F}_\bullet \mapsto \mathcal{F}_\bullet^\vee \quad \text{endows} \quad (dgPerf(X), \mathscr{Q})$$

with a structure of a dg category with weak equivalences and duality, weak equivalences being the set of all quasi-isomorphism $\mathscr{Q}$.

The following proposition on derived equivalences is derived from results in [TT].

Proposition 4.5. *Let X be noetherian separated scheme, with an ample family of line bundles and $k \geq 0$ be an integer. Let $I_\bullet$ be as in Notation 4.3 (2). Then,*

$$\mathscr{D}^k(\mathscr{V}(X)) \to \mathscr{D}^k Perf(X)$$

is an equivalence of derived categories.

Of our particular interest would be the following equivalences of dg categories.

Proposition 4.6. Suppose X is a quasi-projective scheme over an affine scheme Spec (A) and $k \geq 0$ is an integer. Then,

1. The inclusion functor

$$dg^k\mathscr{V}(X) \hookrightarrow dg\mathcal{P}erf^k(X)$$

is a duality preserving form functor (see [S3, 1.12, 1.7], for definition), of pointed dg categories with weak equivalences and dualities, such that the associated functor of the triangulated categories

$$\mathscr{T}\left(dg^k\mathscr{V}(X)\right) \hookrightarrow \mathscr{T}\left(dg\mathcal{P}erf^k(X)\right)$$

is an equivalence.

2. The inclusion functor

$$dgC\mathbb{M}^k(X) \hookrightarrow T^k\left(dgPerf^k(X)\right)$$

is a duality preserving form functor, of pointed dg categories with weak equivalences and dualities, where T denotes the shift. Further, the associated functor of the triangulated categories

$$\mathscr{T}(dgC\mathbb{M}^k(X)) \to \mathscr{T}\left(T^k\left(dgPerf^k(X)\right)\right)$$

is an equivalence.

The following useful diagram is analogous to the diagram in the Equivalence Theorem 2.2, in the context of dg categories with weak equivalences and dualities.

Corollary 4.7. Suppose X is a quasi-projective scheme over an affine scheme Spec (A), and $k \geq 0, r$ are integers. Consider the diagram

$$\begin{array}{ccccc} T^{-1}dgC\mathbb{M}^{k+1}(X) & \longrightarrow & T^k dg\mathcal{P}erf^{k+1}(X) & \longleftarrow & T^k dg^{k+1}\mathscr{V}(X) \\ & & \downarrow & & \downarrow \\ dgC\mathbb{M}^k(X) & \longrightarrow & T^k dg\mathcal{P}erf^k(X) & \longleftarrow & T^k dg^k\mathscr{V}(X) \end{array} \tag{4}$$

In this diagram, all the arrow are form functors of dg categories with weak equivalence. Further, the horizontal arrows induce equivalences of the associated triangulated categories and the right hand square commutes. Note that there is no natural vertical functor on the left side.

Proof. Follows from Proposition 4.6. ∎

Theorem 4.8. *Suppose X is a quasi-projective scheme over an affine scheme* Spec (A)*, with $1/2 \in A$ and $k \geq 0, r$ are integers. In the following, weak equivalences and dualities in the respective categories would be as in (4.6). Then, the maps in the following zig-zag sequences*

$$\mathbf{GW}^{[r]}\left(dgC\mathbb{M}^k(X)\right) \xrightarrow{\zeta} \mathbf{GW}^{[k+r]}\left(dgPerf^k(X)\right) \xleftarrow{\Phi} \mathbf{GW}^{[k+r]}\left(dg^k\mathscr{V}(X)\right) \qquad \text{in Sp}$$

$$\mathbb{G}W^{[r]}\left(dgC\mathbb{M}^k(X)\right) \xrightarrow[\zeta]{} \mathbb{G}W^{[k+r]}\left(dgPerf^k(X)\right) \xleftarrow[\Phi]{} \mathbb{G}W^{[k+r]}\left(dg^k(\mathscr{V}(X)\right) \qquad \text{in BiSp}$$

*are stable homotopy equivalences in the respective categories (*Sp *and* Bi*Sp).*

Proof. Follows from Proposition 4.6 and [S3, Theorem 6.5], [S3, Theorem 8.9]. ∎

Main Theorem in GW-Theorey:

Theorem 4.9. *Suppose X, k, r are as in (4.8). Assume further that X is a Cohen-Macaulay scheme. Consider the following diagram of $\mathbb{G}W$-Bispetra:*

$$\begin{array}{ccccc}
\mathbb{G}W^{[-1+r]}\left(dgC\mathbb{M}^{k+1}(X)\right) & & \mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}(X)\right) & & \coprod_{x\in X^{(k)}}\mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}(X_x)\right) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{G}W^{[k+r]}\left(dgPerf^{k+1}(X)\right) & & \mathbb{G}W^{[k]}\left(dgPerf^{k}X)\right) & & \coprod_{x\in X^{(k)}}\mathbb{G}W^{[k+r]}\left(dgPerf(X_x)\right) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{G}W^{[k+r]}\left(dg^{k+1}\mathscr{V}(X)\right) & \longrightarrow & \mathbb{G}W^{[k+r]}\left(dg^{k}\mathscr{V}(X)\right) & \longrightarrow & \coprod_{x\in X^{(k)}}\mathbb{G}W^{[k+r]}\left(dg^{k}\mathscr{V}(X_x)\right)
\end{array}$$

In this diagram, all the vertical arrows are equivalence of homotopy Bispectra and the bottom sequence is a homotopy fibration of bispectra. In particular, there is a sequence zig-zag maps of Bispectra

$$\mathbb{G}W^{[-1+r]}\left(dgC\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}(X)\right) \longrightarrow \coprod_{x\in X^{(k)}}\mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}(X_x)\right)$$

that is a homotopy fibration.

Proof. It follows directly from Theorem 4.8 that the vertical rows are equivalences. It remains to show that the bottom row is a fibration. This follows from the localization theorem [S3, Thm 8.10] and Lemma 3.1. When X is regular, use the localization theorem [S3, Thm 6.6]. The proof is complete. ■

A remark:

Remark 4.10. The following are some remarks:

1. As in Corollary $\mathbb{K}$-theory, each shift r, corresponding to the fiber sequence in Theorem 4.9, long exact sequence of $\mathbb{G}W_i$-groups would follow.

2. For a scheme X and a rank one locally free sheaf $\mathcal{L}$, $\mathcal{H}om(-, \mathcal{L})$ induces a duality on $\mathscr{V}(X)$. All of the above would be valid, with dualities induced by $\mathcal{H}om(-, \mathcal{L})$, instead of $\mathcal{H}om(-, \mathcal{O}_X)$.

4.1 Comparison with the GW-theory of $C\mathbb{M}^k(X)$

$C\mathbb{M}^k(X)$ being an Exact category with duality (and "isomorphisms"), it has its own $GW\left(C\mathbb{M}^k(X)\right)$ space and hence the groups

$$GW_i\left(C\mathbb{M}^k(X)\right) := \pi_i\left(GW\left(C\mathbb{M}^k(X)\right)\right) \qquad \forall \quad i \geq 0$$

To make better sense out of Theorem 4.9, we compute the groups

$$\mathbb{G}W_i^{[r]}\left(C\mathbb{M}^k(X)\right) := \pi_i\left(\mathbb{G}W^{[r]}\left(dgC\mathbb{M}^k(X)\right)\right) \qquad \forall \quad i \in \mathbb{Z}.$$

Lemma 4.11. For a noetherian (Cohen-Macaulay) quasi-projective scheme X, we have

$$\mathbb{G}W_i^{[0]}\left(dgC\mathbb{M}^k(X)\right) = \\ = \begin{cases} GW_i\left(C\mathbb{M}^k(X)\right) & i \geq 0 \\ \mathbf{GW}\left(\widetilde{S^{-i}C\mathbb{M}^k}(X)\right) =: \mathbb{G}W_i(C\mathbb{M}^k(X)) & i \leq -1 \end{cases}$$

and

$$\mathbb{G}W_i^{[2]}\left(dgC\mathbb{M}^k(X)\right) = \\ = \begin{cases} GW_i\left(C\mathbb{M}^k(X)^-\right) & i \geq 0 \\ \mathbf{GW}\left(\widetilde{S^{-i}C\mathbb{M}^k}(X)^-\right) =: \mathbb{G}W_i(C\mathbb{M}^k(X)^-) & i \leq -1 \end{cases}$$

where "widetilde" indicates idempotent completion. *For analogy with $\mathbb{K}$-theory, see [S1, §2.4.3].*

Thank You!

References

[B1] Balmer, Paul Triangular Witt groups. I. The 12-term localization exact sequence. *K-Theory* 19 (2000), no. 4, 311-363.

[B2] Balmer, Paul Supports and filtrations in algebraic geometry and modular representation theory. Amer. J. Math. 129 (2007), no. 5, 1227-1250.

[B3] Balmer, Paul Niveau spectral sequences on singular schemes and failure of generalized Gersten conjecture. *Proc. Amer. Math. Soc.* 137 (2009), no. 1, 99-106.

[BW] Balmer, Paul; Walter, Charles A Gersten-Witt spectral sequence for regular schemes. *Ann. Sci. École Norm. Sup.* (4) 35 (2002), no. 1, 127?152.

[BH] Bruns, Winfried; Herzog, Jürgen Cohen-Macaulay rings. Cambridge Studies in Advanced Mathematics, 39. *Cambridge University Press, Cambridge*, 1993. xii+403 pp.

[H] Hovey, Mark Spectra and symmetric spectra in general model categories. *J. Pure Appl. Algebra* 165 (2001), no. 1, 63-127.

[G] Gille, Stefan A graded Gersten-Witt complex for schemes with a dualizing complex and the Chow group. *J. Pure Appl. Algebra* 208 (2007), no. 2, 391?419.

[K] Keller, Bernhard On the cyclic homology of exact categories. *J. Pure Appl. Algebra* 136 (1999), no. 1, 1-56.

[MS1] Satya Mandal and Sarang Sane Stability of locally CMFPD homologies under duality *Journal of Algebra* 440 (2015) 49-71

[MS2] Satya Mandal and Sarang Sane, On Dévissage for Witt groups, arXiv:1306.3533

[M1] Mandal, Satya Derived Witt group formalism. *J. Pure Appl. Algebra* 219 (2015), no. 8, 3518-3534.

[M2] Satya Mandal, Sublagrangian constructions, *J. Ramanujan math. Soc*, . 30, No.3 (2015) 309-330

[M3] Mandal, Satya Foxby Morphism and derived equivalences, *J. of Algebra* 440 (2015) 113-127.

[M] Matsumura, Hideyuki Commutative algebra. Second edition. Mathematics Lecture Note Series, 56. *Benjamin/Cummings Publishing Co., Inc., Reading, Mass.*, 1980. xv+313 pp.

[SS] Sarang Sane and William Sanders, Finite homological dimension and a derived equivalence, arXiv:1406.0593

[Q] Quillen, Daniel Higher algebraic K-theory. I. *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pp. 85-147. Lecture Notes in Math., Vol. 341, *Springer, Berlin* 1973.

[RS] Roberts, Paul C.; Srinivas, V. Modules of finite length and finite projective dimension. *Invent. Math.* 151 (2003), no. 1, 1-27.

[S1] Schlichting, Marco Higher algebraic K-theory. *Topics in algebraic and topological K-theory*, 167-241, Lecture Notes in Math., 2008, *Springer, Berlin*, 2011.

[S2] Schlichting, Marco Negative K-theory of derived categories. *Math. Z.* 253 (2006), no. 1, 97-134.

[S3] Schlichting, Marco HERMITIAN K-THEORY, DERIVED EQUIVALENCES AND KAROUBI'S FUNDAMENTAL THEOREM, arXiv:1209.0848

[S4] Reviewed Schlichting, Marco The Mayer-Vietoris principle for Grothendieck-Witt groups of schemes. *Invent. Math.* 179 (2010), no. 2, 349-433.

[TT] Thomason, R. W.; Trobaugh, Thomas Higher algebraic K-theory of schemes and of derived categories. *The*

Grothendieck Festschrift, Vol. III, 247-435, Progr. Math., 88, *Birkhäuser Boston, Boston, MA*, 1990.

[W] Waldhausen, Friedhelm Algebraic K-theory of spaces. *Algebraic and geometric topology* (New Brunswick, N.J., 1983), 318-419, Lecture Notes in Math., 1126, *Springer, Berlin*, 1985.