The Complete Intersection conjecture of Murthy and epimorphism conjecture of Abhyankar

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1 Background and Main Results

We start with the following theorem of Mohan Kumar:

Theorem 1.1. [Mohan Kumar, [Mk]] Suppose $A = R[X]$ is a polynomial ring over a noetherian commutative ring R. Suppose I is an ideal in A that **contains a monic polynomial**.

Assume, $\mu\left(\frac{I}{I^2}\right) \geq \dim\left(\frac{A}{I}\right)+2$ Then, $\exists$ a surjective map $P \twoheadrightarrow I$ where P is a projective A-module with $rank(P) = \mu\left(\frac{I}{I^2}\right)$.

In particular, suppose $A = k[X_1, \ldots, X_n]$ is a polynomial ring over a field k and I is an ideal in A.

$$\text{Assume} \quad \mu\left(\frac{I}{I^2}\right) \geq \dim\left(\frac{A}{I}\right) + 2 \quad \text{Then,} \quad \mu(I) = \mu\left(\frac{I}{I^2}\right)$$

Subsequently, I proved the following:

Theorem 1.2 (Mandal [M9])**.** Suppose $A = R[X]$ is a polynomial ring over a noetherian commutative ring R. Suppose I is an ideal in A that **contains a monic polynomial**.

Assume $\mu\left(\frac{I}{I^2}\right) \geq \dim(A/I) + 2$ Then, $\mu(I) = \mu\left(\frac{I}{I^2}\right)$

In fact, the above were partial answers to the following two versions of Complete Intersection Conjecture of Murthy ([M, M8]):

Conjecture 1.3. There are two versions:

1. **([M, M8]):** Suppose $A = k[X_1, \ldots, X_n]$ is a polynomial ring over a field k. Then,
$$\mu(I) = \mu\left(\frac{I}{I^2}\right)$$
2. (**folklore**): Suppose $A = R[X]$ is a polynomial ring over a noetherian commutative ring R (**regular?**). Suppose I is an ideal in A that **contains a monic polynomial**. Then,
$$\mu(I) = \mu\left(\frac{I}{I^2}\right)$$

Observe, Conjecture (2) $\Longrightarrow$ Conjecture (1)

The following was proved in the recent past:

Theorem 1.4 (Mandal [M5])**.** Let R be a regular ring containing an infinite field k, with $1/2 \in k$. Assume R is essentially smooth over k or k is perfect. Suppose $A = R[X]$ is the polynomial ring and I is an ideal in A that **contains a monic polynomial**.

$$\text{Then,} \qquad \mu(I) = \mu\left(\frac{I}{I^2}\right)$$

In fact, any set of n-generators of I/I^2 lifts to a set of generators of I, when $n \geq 2$.

In particular, Murthy's conjecture is settled, in most cases:

Corollary 1.5 (Mandal)**.** Suppose $A = k[X_1, X_2, \ldots, X_n]$ is a polynomial ring over an infinite field k, with $1/2 \in k$. Suppose I is an ideal in A.

$$\text{Then,} \quad \mu(I) = \mu\left(\frac{I}{I^2}\right)$$

Remark: When k is infinite perfect, Fasel proved this result (1.5) with significant contributions from me, (*acknowledged to an extent*, e. g. [F, Lemma 3.1.2]). I collaborated with him for four weeks, during my trip to him in May-June 2015, under the assumption that we were jointly working.

Before we proceed, let me recall the two versions of S. Abhyankar's epi-morphism conjecture [DG], which has been a folklore, in most part.

Conjecture 1.6 (S. Abhyankar)**.** Let k be a field with $\mathbb{Q} \subseteq k$.

$$\text{Suppose } \varphi : k[X_1, X_2, \ldots X_n] \twoheadrightarrow k[Y_1, Y_2, \ldots Y_m]$$

is an **epimorphism** of polynomial k-algebras and $I = \ker(\varphi)$. Then, it is conjectured (folklore)

1. that I is generated by variables. That means, $I = (X_1', X_2', \ldots, X_{n-m}')$ and
$$k[X_1, X_2, \ldots X_n] = k[X_1', X_2', \ldots X_{n-m}', \ldots, X_n']$$
2. (**The weaker version**): that
$$\mu(I) = \mu\left(\frac{I}{I^2}\right)$$

While Abhyankar's epimorphism conjecture remains a folklore, in most part, Amartya Dutta and Neena Gupta [DG] published a very helpful survey recently. As indicated in [DG], very limited progress has been made on either version of Abhyankar's epimorphism conjectures (arguably) (if anyone of us could solve this one, that would be worthwhile.) However, the weaker version of S. Abhyankar's epi-morphism conjecture 1.6 follows from (1.4), as follows, This is significant, because of the same reason.

Theorem 1.7 (Mandal)**.** Let R be a regular ring over an infinite field k, with $1/2 \in k$. Assume R is essentially smooth over k or k is perfect. Suppose

$$\varphi : R[X_1, X_2, \dots X_n] \twoheadrightarrow R[Y_1, Y_2, \dots Y_m] \quad \text{is an epimorphism}$$

of polynomial R-algebras and $I = \ker(\varphi)$. If $n - m \geq \dim R + 1$,

$$\text{Then,} \quad \mu(I) = \mu\left(\frac{I}{I^2}\right)$$

In particular, if R is local, then I is a complete intersection ideal.

Proof. Regarding existence of monic polynomials in I, see Suslin's theorem in my book [M8, pp. 78]. ∎

The weaker version of S. Abhyankar's epi-morphism conjecture [DG] is settled affirmatively, for infinite fields k, with $1/2 \in k$, as follows.

Corollary 1.8 (Mandal)**.** Suppose k is an infinite field, with $1/2 \in k$. Suppose

$$\varphi : k[X_1, X_2, \dots X_n] \twoheadrightarrow k[Y_1, Y_2, \dots Y_m] \quad \text{is an epimorphism}$$

of polynomial k-algebras and $I = \ker(\varphi)$.

$$\text{Then,} \quad \mu(I) = \mu\left(\frac{I}{I^2}\right) = n - m$$

Proof. From properties of regular rings, I is local complete intersection ideal (see [CA, pp. 121]). Therefore, $\frac{I}{I^2}$ is projective and hence free. Now this follows from (1.4). ∎

Comments and Open Problems:

1. Prove Murthy's conjectures (1) when $char(k) = 2$
2. Prove Murthy's conjectures (1) when $char(k) \neq 2$ and k is finite. Marco Schlichting told me that this is a matter of working out.
3. Prove Murthy's conjectures (1) when $k = \mathbb{Z}$.
4. I did not study Abhyankar's Epimorphism conjecture 1.6 in the recent past. I know the following:
   (a) It fails when $char(k) \neq 0$ (see [A]).
   (b) (see [A, Cor 9.21, pp. 75]) It works for epimorphisms
   $$k[X, Y] \twoheadrightarrow k[Z]$$
   Then, Epimorphism conjecture 1.6 is valid.
   (c) (See [DG, §2.3]) Suppose k is algebraically closed field, with $char(k) = 0$ and
   $$\varphi : k[X_1, \ldots, X_n] \to k[Y_1, \ldots, Y_m]$$
   is an epimorphism. Let $I = \ker(\varphi)$. Assume $n \geq 2m + 2$. Then, Epimorphism conjecture 1.6 is valid.
   (d) **Proof Rest of Epimorphism Conjecture 1.6:** **Job of your generation, unless one of us finish it. I will be happy to work with anyone on this one.**

2 Method of proofs: Homotopy and Monic polynomials

The idea of Homotopy was around to deal with Murthy's conjectures, for a while, which was originally introduced by Nori ([M10]). The same were used in [F, M5].

We start with Nori's Homotopy conjecture:

Conjecture 2.1 (M. V. Nori)**.** Suppose $X = \mathrm{Spec}\,(A)$ is a smooth affine scheme over a field k and P is a projective A-module of rank r. Suppose $f_0 : P \twoheadrightarrow I_0$ is a surjective homomorphism, where I_0 is an ideal of A. Now suppose, $I \subseteq A[T]$ is an ideal in the polynomial ring $A[T]$ such that $I(0) = I_0$ and $\varphi : P \otimes A[T] \twoheadrightarrow \frac{I}{I^2}$ is a surjective map, such that φ is compatible with f_0. Then, there is a surjective homomorphism $\psi : P \otimes A[T] \twoheadrightarrow I$ such that $\psi_{|T=0} = f_0$ and ψ lifts φ.

We interpret, Nori's Homotopy conjecture when $P = A^n$ is free:

1. Let $I_0 \subseteq A$ and $I \subseteq A[T]$ is ideals in the respective rings, such that $I(0) = I_0$.
2. Let $n \geq 0$ is an integer. Assume,
$$I_0 = (a_1, \ldots, a_n), \quad I = (f_1(T), \ldots, f_n(T)) + I^2$$
$$\ni \qquad f_i(0) \equiv a_i \mod \mathrm{I}_0^2$$
3. Then, the conjecture is
$$I = (F_1(T), \ldots, F_n(T)) \ \ni \ F_i(0) = a_i, \ F_i(T) \equiv f_i(T) \mod \mathrm{I}^2$$

Homotopy is an age old concept, and we give the following definitions:

Definition 2.2. Suppose A is a commutative noetherian ring and P is a projective A-module and I is an ideal of A. A surjective homomorphism $f : P \twoheadrightarrow \frac{I}{I^2}$ would be called a P-local orientation.

$$\text{Let} \quad \mathrm{f_0, f_1 : P \twoheadrightarrow \frac{I_i}{I_i^2}} \quad \text{be two P} - \text{local orientations.}$$

We say that f_0 is (strictly) homotopic to f_1, if there is a $P[T]$-local orientation

$$F : P[T] \twoheadrightarrow \frac{I}{I^2} \ \ni \ F(0) = f_0 \text{ and } F(1) = f_1$$

Consider the equivalence relation generated by strict homotopy. We say, f_0 is homotopic to f_1, if they are equivalent to each other. (Analyze the case when $P = A^n$ is free.)

A relaxed version of Nori's Homotopy conjecture 2.1 is:

Conjecture 2.3. Use the notations as in (2.3).

$$\text{Suppose} \quad f_0, f_1 : P \twoheadrightarrow \frac{I_i}{I_i^2} \quad \text{two P local orientations.}$$

Assume f_0 is (strictly) homotopic to f_1.

$$\text{Suppose} \quad \exists \text{ surjective map } \varphi_0 \ \ni \quad \begin{tikzcd} & I_0 \arrow[d] \\ P \arrow[ur, dashed, two heads, "\varphi_0"] \arrow[r, two heads, "f_0"'] & \frac{I_0}{I_0^2} \end{tikzcd} \quad \text{commutes.}$$

Then, same is true about f_1.

(Analyze the case when $P = A^n$ is free.)

Remark. I will show soon, that the solutions of Conjecture 2.3, would settle the conjectures of Murthy (1):

1. There are two trivial ideals, $I_0 = A$ or $I_0 = (0)$. In both cases, $\frac{I_0}{I_0^2} = 0$. Any set of n generators of $\frac{I_0}{I_0^2}$, lift to generator of I_0.
2. If Conjecture 2.3 is valid, to prove Murthy's conjecture (1), we will prove that the set of generators are homotopic to this trivialities.

3. When I arrived in Grenoble to visit Jean Fasel [F], in May 2015, he proposed we work on Conjecture 2.3, when P is free, with a goal to prove Murthy's conjecture.

   I immediately told him that this is only a version of Nori's Homotopy conjecture [M10]. Because of my faith in the **invisibility of monic polynomials**, it did not take too long for me to figure out the following proposition.

   Proposition 2.4. Suppose $A = R[X]$ is a polynomial ring over a commutative ring R and I_1 is an ideal
   that contains a monic polynomial.
   Suppose $\omega : A^n \twoheadrightarrow \frac{I_1}{I_1^2}$ is a surjective homomorphism (*local orientation*).
   Then, ω is (strictly) homotopic to $A^n \twoheadrightarrow \frac{A}{A^2}$.
   (*Which of course lifts to generators of $I_0 := A$.*)

   Proof. Postpone!

3 The Obstruction presheaf

We will make some of the above more funtorial, **which is not deep**.

There are skeptics and enthusiasts regarding $\mathbb{A}^1$-homotopy theory. Lately, I probed into it. However, I am convinced that there is a new way to look at things, while I am not competent to say if it really cracks anything. I understood that they try to look at everything as functors or presheafs, which has some advantages. That is why we would restructure the above definition of homotopy. **This is not deep**. We will not talk about $\mathbb{A}^1$-homotopy.

First, we establish some notations that will be useful throughout these talks.

Notations 3.1. Throughout, k will denote a field (or ring), with $1/2 \in k$ and A, R will denote commutative noetherian rings. For a commutative ring A and a finitely generated A-module M, the minimal number of generators of M will be denoted by $\mu(M)$.

We denote

$$q_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z^2, \qquad \tilde{q}_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z(Z-1).$$

Denote

$$Q_{2n} = \operatorname{Spec}(\mathscr{A}_{2n}) \text{ where } \mathscr{A}_{2n} = \frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]}{(\tilde{q}_{2n+1})} \tag{1}$$

and

$$Q'_{2n} = \operatorname{Spec}(\mathscr{B}_{2n}) \text{ where } \mathscr{B}_{2n} = \frac{k[X_1, \ldots, X_n, Y_1, \ldots, Y_n, Z]}{(q_{2n+1} - 1)}. \tag{2}$$

There are inverse isomorphisms

$$\alpha : \mathscr{A}_{2n} \xrightarrow{\sim} \mathscr{B}_{2n} \qquad \beta : \mathscr{B}_{2n} \xrightarrow{\sim} \mathscr{A}_{2n}$$

given by

$$\begin{cases} \alpha(x_i) = \frac{x_i}{2} & 1 \leq i \leq n \\ \alpha(y_i) = \frac{y_i}{2} & 1 \leq i \leq n \\ \alpha(z) = \frac{z+1}{2} \end{cases} \qquad \begin{cases} \beta(x_i) = 2x_i & 1 \leq i \leq n \\ \beta(y_i) = 2y_i & 1 \leq i \leq n \\ \beta(z) = 2z - 1 \end{cases} \tag{3}$$

Therefore, $Q_{2n} \cong Q'_{2n}$.

Definition 3.2. The category of schemes over $\mathrm{Spec}\,(k)$ will be denoted by $\underline{\mathrm{Sch}}_k$. Also, $\underline{\mathrm{Sets}}$ will denote the category of sets.

Given a scheme $Y \in \underline{\mathrm{Sch}}_k$, the association $X \mapsto \mathcal{H}om(X, Y)$ is a presheaf on $\underline{\mathrm{Sch}}_k$. (**Recall, a presheaf is a contravariant functor**.)

This presheaf is often identified with Y, itself. So, in some literature one may write, Y for the presheaf $\mathcal{H}om(-, Y)$ and $Y(X) := \mathcal{H}om(X, Y)$.

With such an approach, for $X = \mathrm{Spec}\,(A)$, it follows immediately that, $Q_{2n}(A)$ and $Q'_{2n}(A)$ can be identified with the sets, as follows:

$$Q_{2n}(A) = \left\{ (f_1, \ldots, f_n; g_1, \ldots, g_n; s) \in A^{2n+1} : \sum_{i=1}^{n} f_i g_i + s(s-1) = 0 \right\}$$

$$Q'_{2n}(A) = \left\{ (f_1, \ldots, f_n; g_1, \ldots, g_n; s) \in A^{2n+1} : \sum_{i=1}^{n} f_i g_i + s^2 - 1 = 0 \right\}$$

The homotopy pre-sheaves are given by the pushout diagrams in Sets:

$$\begin{array}{ccc} Q_{2n}(A[T]) & \xrightarrow{T=0} & Q_{2n}(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ Q_{2n}(A) & \longrightarrow & \pi_0\left(Q_{2n}\right)(A) \end{array} \quad \text{and} \quad \begin{array}{ccc} Q'_{2n}(A[T]) & \xrightarrow{T=0} & Q'_{2n}(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ Q'_{2n}(A) & \longrightarrow & \pi_0\left(Q'_{2n}\right)(A) \end{array}$$

The isomorphism $Q_{2n} \cong Q'_{2n}$, induces
a bijection $\pi_0\left(Q_{2n}\right)(A) \cong \pi_0\left(Q'_{2n}\right)(A)$.

Remark for mature audience: By including more variables, you can define higher homotopy sets/sheaves $\pi_i\left(Q_{2n}\right)(A)$. In fact you do the same for any contravariant functor.

For any ring A and

$$\mathbf{v} = (f_1, \ldots, f_n; g_1, \ldots, g_n; s) \in Q_{2n}(A), \ let\ I(\mathbf{v}) := (f_1, \ldots, f_n, s)A$$

Also, let $\omega_{\mathbf{v}} : A^n \to \frac{I(\mathbf{v})}{I(\mathbf{v})^2}$ denote the surjective homomorphism defined by $e_i \mapsto f_i + I^2$ where $e_1, \ldots, e_n$ is the standard basis of A^n.

Definition 3.3. Suppose A is a commutative ring and I is an ideal in A. For an integer $n \geq 1$, and a local A^n-local orientation, $\omega : A^n \twoheadrightarrow I/I^2$, would be called a local n-orientation of I (**renaming**).

Let $\mathcal{O}(A, n) = \left\{ (I, \omega) : \omega : A^n \twoheadrightarrow \dfrac{I}{I^2} \quad \text{is a local n} - \text{orientation} \right\}$

For $(I, \omega) \in \mathcal{O}(A, n)$, write

$$\zeta(I, \omega) := [(f_1, \ldots, f_n; g_1, \ldots, g_n, s)] \in \pi_0\left(Q_{2n}(A)\right)$$

where

1. $\omega : A^n \twoheadrightarrow \frac{I}{I^2}$ is given by $e_i \mapsto f_i + I^2$
2. and hence $\sum_{i=1}^n f_i g_i + s(s-1) = 0$ for some $g_1, \ldots, g_n \in A$ and $s \in I$.
3. Note,
$$(f_1, \ldots, f_n; g_1, \ldots, g_n; s) \in Q_{2n}(A).$$

It was established in [F, Theorem 2.0.7], that this association is well defined. We refer to $\zeta(I, \omega)$, as an obstruction class. Therefore, we have a commutative diagram

$$\begin{tikzcd} Q_{2n}(A) \arrow[d, two heads, "\eta"'] \arrow[dr, "\zeta"] & \\ \mathcal{O}(A, n) \arrow[r, dashed, "\zeta"'] & \pi_0\left(Q_{2n}(A)\right) \end{tikzcd}$$

and $\eta(\mathbf{v}) = (I(\mathbf{v}), \omega_{\mathbf{v}})$. Note that we use the same notation ζ for two set theoretic maps.

We comment

1. Note that $Q_{2n}(A) \equiv Hom(A, Q_{2n})$ is a presheaf, while $\mathcal{O}(n, A)$ is not. **This why $Q_{2n}(A)$ wins**, and we want to work with it, instead of $\mathcal{O}(n, A)$.

2. We define $\mathbf{u}, \mathbf{v} \in Q_{2n}(A)$ homotopic, if they have same images in $\pi_0\left(Q_{2n}\right)(A)$.

   Define $\mathbf{u}, \mathbf{v} \in Q_{2n}(A)$ to be strictly homotopic, if

$$\exists\ F(T) \in Q_{2n}(A[T]) \ \ni\ F(0) = \mathbf{u},\ F(1) = \mathbf{v}$$

Proposition 3.4. Suppose $A = R[X]$ is a polynomial ring over a commutative ring R and I is an ideal that contains a monic polynomial. Suppose $\omega : A^n \twoheadrightarrow I/I^2$ is a surjective homomorphism (*local orientation*). Then $\zeta(I, \omega) = [\mathbf{0}] \in \pi_0(Q_{2n})(A)$, where $\mathbf{0} := (1, 0, \ldots, 0; 0, \ldots, 0; 0) \in Q_{2n}(A)$.

Proof. Let $f_1, \ldots, f_n \in I$ be a lift of ω. Then,

$$I = (f_1, f_2, \ldots, f_n) + I^2$$

We can assume that f_1 is a monic polynomial, with even degree. Now, consider the transformation [M9]:

$$\varphi : R[X, T^{\pm 1}] \xrightarrow{\sim} R[X, T^{\pm 1}] \quad \text{by} \quad \begin{cases} \varphi(X) = X - T + T^{-1} \\ \varphi(T) = T \end{cases}$$

There is a commutative diagram

$$\begin{tikzcd} R[X] \arrow[r, equal] \arrow[d] & R[X] \\ R[X, T^{\pm 1}] \arrow[r, "\varphi"'] & R[X, T^{\pm 1}] \arrow[u, "T=1"'] \end{tikzcd}$$

Then, $\varphi(f_1) = f_1(X - T + T^{-1})$ is doubly monic in T, meaning that its lowest and the highest degree terms have coefficients 1. Let $F_1(X, T) = T^{\deg f_1(X)} \varphi(f_1) \in R[X, T]$. Then, $F_1(X, 0) = 1$. Also, for $i = 2, \ldots, n$ write $F_i(X, T) = T^{\delta} \varphi(f_i)$, for some integer $\delta \gg 0$, such that $F_i(X, T) \in TR[X, T]$. Therefore, $F_i(X, 0) = 0$. Now, write

$$\mathscr{I}' = \varphi(IR[X, T^{\pm 1}]) \quad \text{and} \quad \mathscr{I} := \mathscr{I}' \cap R[X, T].$$

Since $\frac{R[X,T]}{\mathscr{I}} \xrightarrow{\sim} \frac{R[X,T^{\pm 1}]}{\mathscr{I}'}$, it follows

$$\mathscr{I} = (F_1(X,T), \ldots, F_n(X,T)) + \mathscr{I}^2.$$

Therefore, by Nakayama's Lemma, there is a $S(X,T) \in \mathscr{I}$, such that

$$(1 - S(X,T))\mathscr{I} \subseteq (F_1(X,T), F_2(X,T), \ldots, F_n(X,T)).$$

and hence

$$\sum F_i(X,T)G_i(X,T) + S(X,T)(S(X,T) - 1) = 0$$

for some $G_1, \ldots, G_n \in A[X,T]$. Write $\psi(X,T) =$

$$(F_1(X,T), F_2(X,T), \ldots, F_n(X,T); G_1(X,T), \ldots, G_n(X,T); S(X,T))$$

Then, $\psi(X,T) \in Q_{2n}(R[X,T])$ and $\mathscr{I}_{|T=1} = I$. Further,

$$\psi(X,1) = (f_1, \ldots, f_n; G_1(X,1), \ldots, G_n(X,1); S(X,1))$$

and

$$\psi(X,0) = (1, 0, \ldots, 0; G_1(X,0), \ldots, G_n(X,0), S(X,0)).$$

By [F, 2.0.10], $\psi(X,0) \sim \mathbf{0} \in Q_{2n}(A)$. Hence, $\psi(X,1) \sim \mathbf{0} \in Q_{2n}(A)$. Therefore,

$$\zeta(I,\omega) = [\psi(X,1)] = [\mathbf{0}] \in \pi_0\left(Q_{2n}(R)\right).$$

The proof is complete. ■

Remark 3.5. *In the light of (3.4), our objective would be to prove if* $\mathbf{v} \in Q_{2n}(A)$ *is homotopically trivial, then the corresponding local* n*-orientation*

$$\omega_{I_{\mathbf{v}}} : A^n \twoheadrightarrow \frac{I_{\mathbf{v}}}{I_{\mathbf{v}}^2} \quad \text{lifts to a surjection} \quad \begin{tikzcd} A^n \arrow[r, dashed, two heads] \arrow[dr, two heads, "\omega_{I_{\mathbf{v}}}"'] & I_{\mathbf{v}} \arrow[d, two heads] \\ & \frac{I_{\mathbf{v}}}{I_{\mathbf{v}}^2} \end{tikzcd}$$

4 Homotopy and the lifting property

Recall, for any commutative ring A, we have $\mathcal{A}_{2n}(A) \xrightarrow{\sim} \mathcal{B}_{2n}(A)$. Therefore, there are set theocratic bijections

$$\alpha : Q_{2n}(A) \xrightarrow{\sim} Q'_{2n}(A), \quad \beta : Q'_{2n}(A) \xrightarrow{\sim} Q_{2n}(A)$$

This induces, set theocratic bijections

$$\pi_0\left(Q_{2n}\right)(A) \xrightarrow{\sim} \pi_0\left(Q'_{2n}\right)(A)$$

4.1 Elementary Orthogonal group and Lifting

Some Prelude to the Elementary Orthogonal groups

1. Recall, $GL_n(A)$ has a subgroup $EL_n(A)$ of elementary matrices.
2. Given any quadratic form q (non-singular), over k of rank n, we can define
   (a) Orthogonal subgroups $O(A^n, q) \subseteq GL_n(A)$, defined same way as we do in Linear Algebra classes.
   (b) In analogy to elementary subgroup, there is a elementary orthogonal subgroups $EO(q, A^n) \subseteq O(A^n, q)$.
3. We will be considering $q_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z^2$.
   (a) The elementary orthogonal subgroup $EO\left(A^{2n+1}, q_{2n+1}\right)$ acts on $Q'_{2n}(A)$.

(b) We Pull this action to $Q_{2n}(A)$, as follows.

Definition 4.1. Define an action on $Q_{2n}(A)$ as follows:

$$\forall\ \mathbf{v} \in Q_{2n}(A), M \in EO\left(A, q_{2n+1}\right) \quad \text{define} \quad \mathbf{v} * M := \beta\left(\alpha(\mathbf{v})M\right)$$

This action is not given by the usual matrix multiplication. Five different classes of the generators of $EO\left(A^{2n+1} q_{2n+1}\right)$ and their actions on $Q_{2n}(A)$ are given in [F].

Theorem 4.2. *Let A be a essentially smooth algebra over an infinite field k, with $1/2 \in k$. Then, for $n \geq 2$, the natural map*

$$\varphi : \frac{Q'_{2n}(A)}{EO(A, q_{2n+1})} \longrightarrow \pi_0\left(Q'_{2n}\right)(A) \quad \text{is a bijection.}$$

Proof. See my paper. When k is infinite perfect, this was proved by Fasel [F]. I extended to the infinite field case. ∎

5 Final Lifting Theorems

I will SKIP most of the following details.

Definition 5.1. *Let A be a commutative ring over k. Let $\mathbf{v} \in Q_{2n}(A)$. We write $\mathbf{v} := (a_1, \ldots, a_n; b_1, \ldots, b_n; s)$. For integers, $r \geq 1$ we say that r-lifting property holds for $\mathbf{v}$, if*

$$I(\mathbf{v}) = (a_1 + \mu_1 s^r, \ldots, a_n + \mu_n s^r) \qquad \text{for some } \mu_i \in A.$$

We say the lifting property holds for $\mathbf{v}$, if

$$I(\mathbf{v}) = (a_1 + \mu_1, \ldots, a_n + \mu_n) \qquad \text{for some } \mu_i \in I(\mathbf{v})^2.$$

The following is the main homotopy invariance theorem, for lifting generators of I/I^2. When k is infinite perfect, theorem was proved in [F], with significant contributions from me.

Theorem 5.2. *Suppose A is a regular ring containing a field k, with $1/2 \in k$. Let $n \geq 2$ be an integer. Let $\mathbf{v} \in Q_{2n}(A)$ and $M \in EO(A, q_{2n+1})$. Then, $\mathbf{v}$ has 2-lifting property if and only if $\mathbf{v} * M$ has the 2-lifting property.*

Proof. We outline the proof. It would be enough to assume that M is a generator of $EO(A, q_{2n+1})$. There would be five cases to deal with, one for each type of generators of $EO(A, q_{2n+1})$, listed in [F, pp 3-4]. Only of them is nontrivial, that is of the case of generators of the type 4 (in the list [F, pp 3-4]). This case follows, mainly from Theorem 5.3 (which [F, Lemma 3.1.2] I gave him in a pdf to the author of [F]). In deed, I spotted the gap in the proof of (see [F, Lemma 3.1.2]), in the first version of [F] and communicated to him, what needs to be done to apply Theorem 5.3. ■

We state the following homotopy lifting theorem [F, Corollary 3.2.6], due to this author (unpublished), that was used crucially in [F].

Theorem 5.3 (Mandal)**.** *Let R be a regular ring containing a field k. Let*
$H(T) := (f_1(T), \ldots, f_n(T), g_1(T), \ldots, g_n(T), s) \in Q_{2n}(R[T])$, with $s \in R$.
Write $a_i = f_i(0), b_i = g_i(0)$. Write $I(T) = (f_1(T), \ldots, f_n(T), s)$. Also assume $I(0) = (a_1, \ldots, a_n)$. Then,

$$I(T) = (F_1, \ldots, F_n) \quad \ni \quad f_i - F_i \in s^2 R[T]$$

Proof. See [F, Lemma 3.1.2], communicated by myself. ■

Remark 5.4. *Note, there is no mention of Homotopy in Theorem 5.2. We will show that homotopy relations reduces to the equivalences defined by the action of $EO(q_{2n+1})$.*

6 Further Details: Homotopy to the Action of $EO(q_{2n+1})$

The following is the quadratic analogue of the result of Ton Vorst [T, pp 507].

Theorem 6.1. *Suppose A is a regular ring containing a field k. Then,*

$$\forall\, \sigma(T) \in O(A[T], q_{2n+1}),\ \ \sigma(0) = 1 \Longrightarrow \sigma(T) \in EO(A[T], q_{2n+1}).$$

Proof. In the case when k is perfect, it follows from the theorem of Stavrova ([S, Theorem 1.3]) on REDUCTIVE groups. I reduce it to the perfect field case, using Popescu's theorem.
I remark that an elementary proof would be possible, without using Stavrova's theorem, exactly as the proof of Ton Vorst ([T, pp 507]). Someone needs to work it out.

The following is repeat of Theorem 4.2.

Theorem 6.2. *Let A be a essentially smooth algebra over an infinite field k, with $1/2 \in k$. Then, for $n \geq 2$, the natural map*

$$\varphi : \frac{Q'_{2n}(A)}{EO(A, q_{2n+1})} \longrightarrow \pi_0\left(Q'_{2n}\right)(A) \quad \text{is a bijection.}$$

Proof. See my paper. According to an expert on quadratic forms, this lemma is standard, which I am not surprised, because of the structure of the proof. ∎

The following summarizes the final results on homotopy and lifting of generators (also see [F, Theorem 3.2.7]).

Theorem 6.3 (Mandal)**.** Suppose A is a regular ring containing an infinite field k, with $1/2 \in k$. Assume A is essentially smooth over k or k is perfect. Let $n \geq 2$ be an integer. Denote $\mathbf{0} := (0, \ldots, 0; 0, \ldots, 0; 0) \in Q_{2n}(A)$ and let $\mathbf{v} \in Q_{2n}(A)$. Then, the following conditions are equivalent:

1. The obstruction $\zeta\left(I(\mathbf{v}, \omega_{\mathbf{v}})\right) = [\mathbf{0}] \in \pi_0\left(Q_{2n}\right)(A)$.
2. $\mathbf{v}$ has 2-lifting property.
3. $\mathbf{v}$ has the lifting property.
4. $\mathbf{v}$ has r-lifting property, $\forall\ r \geq 2$.

Proof. It is clear, (2) $\Longrightarrow$ (3). To prove (3) $\Longrightarrow$ (1), suppose $I(\mathbf{v}) = (a_1 + \mu_1, \ldots, a_n + \mu_n)$, with $\mu_i \in I(\mathbf{v})^2$. Write $\mathbf{v}' = (a_1 + \mu_1, \ldots, a_n + \mu_n; 0, \ldots, 0; 0) \in Q_{2n}(A)$. By [F, 2.0.10], we have $\zeta\left(I(\mathbf{v}, \omega_{\mathbf{v}})\right) = \zeta\left(I(\mathbf{v}', \omega_{\mathbf{v}'})\right) = [v_0] \in \pi_0\left(Q_{2n}\right)$. This establishes, (3) $\Longrightarrow$ (1).

Now we prove (1) $\Longrightarrow$ (2). Assume $\zeta\left(I(\mathbf{v}, \omega_{\mathbf{v}})\right) = [\mathbf{0}]$. In case A is essentially finite over k, it follows from Theorem 6.2 that $\mathbf{0} = \mathbf{v} * M$, for some $M \in EO(A, q_{2n+1})$ and (2) follows from Theorem 5.2. However, when A is regular and contains an infinite perfect field, we have to use Popescu's theorem. By definition, $\zeta\left(I(\mathbf{v}, \omega_{\mathbf{v}})\right) = [\mathbf{0}]$ implies that there is a chain homotopy from $\mathbf{v}$ to $\mathbf{0}$. This data can also be encapsulated in a finitely generated algebra A' over k. As in the proof of (6.1) there is a diagram

$$\begin{tikzcd} k \arrow[r] \arrow[drr] & A' \arrow[r, hook, "\iota"] \arrow[dr] & B \arrow[d] \\ & & A \end{tikzcd} \quad \text{of homomorphisms}$$

such that B is smooth over k. The homotopy relations are carried over to B. Therefore, by replacing A by B, we can assume that A is essentially smooth over k. So, Theorem 6.2 applies and (2) follows as in the previous case.

So, it is established that (1) $\iff$ (2) $\iff$ (3). It is clear that (4) $\Longrightarrow$ (2). Now suppose, one of the first three conditions hold. Fix $r \geq 2$. Notice $I(\mathbf{v}) = (a_1, \ldots, a_n, s^r)A$.

So, replacement of s by s^r leads to the same obstruction class in $\in \pi_0 \left(Q_{2n}\right)(A)$, which is $= [\mathbf{0}] \in \pi_0 \left(Q_{2n}\right)(A)$. Since (1) $\Longleftrightarrow$ (2), it follows $I(\mathbf{v})$ has $2r$-lifting property and hence the r-lifting property. The proof is complete. $\blacksquare$

A Abstract

Abstract: For a commutative ring A and a finitely generated A-module M, we denote $\mu(M) :=$ minimal number of generators of M. It follows, from Nakayama's Lemma, that (**exercise**),

$$\text{for any ideal } I \subseteq A \qquad \mu\left(\frac{I}{I^2}\right) \leq \mu(I) \leq \mu\left(\frac{I}{I^2}\right) + 1$$

The following is the statement of Murthy's complete intersection conjecture:

Conjecture A.1 (M. P. Murthy)**.** Suppose $A = k[X_1, X_2, \ldots, X_n]$ is a polynomial ring over a field k. Suppose I is an ideal in A. Then,

$$\mu(I) = \mu\left(\frac{I}{I^2}\right)$$

This conjecture is settled, affirmatively, when k is an infinite field with $1/2 \in k$. Further, S. Abhyanka's epimorphism conjecture has two versions, as follows:

Conjecture A.2 (S. Abhyankar)**.** Let k be a field with $char(k) = 0$. Suppose

$$\varphi : k[X_1, X_2, \ldots X_n] \twoheadrightarrow k[Y_1, Y_2, \ldots Y_m] \quad \text{is an epimorphism}$$

of polynomial k-algebras and $I = \ker(\varphi)$. Then, it is conjectured (folklore)

1. that I is generated by variables. That means, $I = (X'_1, X'_2, \ldots, X'_{n-m})$ and

$$k[X_1, X_2, \ldots X_n] = k[X'_1, X'_2, \ldots X'_{n-m}, \ldots, X'_n]$$

2. (**The weaker version**): that

$$\mu(I) = \mu\left(\frac{I}{I^2}\right)$$

The weaker version of the epimorphism conjecture follows from Murthy's complete intersection conjecture. I plan to give two or three lectures on these results. Following are come highlights or comments:

1. In the first lecture I will explain the results and would write down the open problems. In the second lecture I will go through the methods.

2. I should be able to write down some exercises that may be suitable for prelims or quals.

3. Anecdotally, my interview talk at KU in 1988, was on the conjecture of Murthy.

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