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On Automorphisms of Modules over Polynomial Rings

S. M. BHATWADEKAR AND SATYA MANDAL*

Tata Institute of Fundamental Research, School of Mathematics, Homi Bhabha Road, Bombay 400 005, India

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1. INTRODUCTION

All rings that we consider in this paper are assumed to be commutative and noetherian. R and A will always denote a ring of this kind. The modules that we consider are also assumed to be finitely generated.

As the title indicates, in this paper we shall be discussing the automorphisms of modules over polynomial rings.

We shall mainly be interested in studying, *for a projective module P over a polynomial ring $R = A[X]$, when is the natural map* $\mathrm{Aut}_R(P) \to \mathrm{Aut}_A(P/XP)$ *surjective*? We may refer to this problem as *the lifting problem.*

The main result (3.1) about the lifting problem is that *if* $R = A[X_1, ..., X_n]$ *is a polynomial ring over A and P is a projective R-module with* rank $P > \dim A$, *then the natural map* $\mathrm{Aut}_R(P) \to \mathrm{Aut}_{A'}(P/X_nP)$ *is surjective, where* $A' = A[X_1, ..., X_{n-1}]$.

In particular, it follows that (3.3) *if* $R = A[X]$ *is a polynomial ring over A and P is a projective R*-module *with* rank $P > \dim A$, *then the natural map* $\mathrm{Aut}_R(P) \to \mathrm{Aut}_A(P/XP)$ *is surjective.*

We also conjecture (3.4) that the condition on the rank of the module is not necessary.

The other type of problem that we study is that, *for an* $R = A[X]$-module M, *how does the subgroup* $SL_2(R)\, EL(R^2 \oplus M)$, *that is generated by* $SL_2(R)$ *and the transvections of* $R^2 \oplus M$, *act on the set of all special* (type of) *unimodular elements of* $R^2 \oplus M$.

This type of study was originally done by Suslin [S2] when M is a free module. In some recent developments, Lindel has proved that [L, 2.8, theorem] *for a projective module P over a polynomial ring* $R =$

* Present address: University of Kansas, Lawrence, KS 66045. Supported by MSRI, Berkeley, CA 94720.

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$A[X_1, ..., X_n]$ *with* rank $P > \dim A$, $EL(R \oplus P)$ *acts transitively on the set of all unimodular elements of* $R \oplus P$.

Our present discussion (Sects. 4, 5, 6) on the action of $SL_2(R)\, EL(R^2 \oplus M)$ was highly inspired by the work of Lindel [L]. In Section 4, we shall make some comments on the work of Lindel and the main result (5.2) will be discussed in Section 5. The main result (5.2) is that *if* M *is an extended* $R = A[X]$-module, *then* $SL_2(R)\, EL(R^2 \oplus M)$ *acts transitively on the set of all special* (type of) *unimodular elements of* $R^2 \oplus M$.

We also conjecture (6.1) that $SL_2(R)\, EL(R^2 \oplus P)$ *acts transitively on the set of all special* (type of) *unimodular elements of* $R^2 \oplus P$, *whenever* P *is a projective* R-module. And the counterexample (6.2) will show that we need to assume that P is projective.

The study of the lifting problem will be done in Sections 2 and 3. The action of $SL_2(R)\, EL(R^2 \oplus M)$ will be discussed in Sections 4, 5, 6.

2. Preliminary Discussions and Notations

As mentioned in the Introduction, the emphasis of our discussions in this paper lies in the "*lifting problem*"; i.e., *if* $R = A[X]$ *is a polynomial ring over a noetherian and commutative ring* A *and if* P *is a projective* R-module, *then is the natural map* $\mathrm{Aut}_R(P) \to \mathrm{Aut}_A(P/XP)$ *surjective*? The main results on this problem will appear in the next section. In this section, we shall do some preliminary discussions and set up some notations.

First we shall record some of the well-known results and immediate comments.

(2.1) *Let* $R = A[X]$ *be a polynomial ring over* A *and let* P *be an extended* R-module. *Then the map* $\mathrm{Aut}_R(P) \to \mathrm{Aut}_A(P/XP)$ *is surjective.*

(2.2) *Let* R *be as in* (2.1) *and let* P *be a projective* R-module *of rank one. Then the map* $\mathrm{Aut}_R(P) \to \mathrm{Aut}_A(P/XP)$ *is surjective.*

(2.3) *Suppose that* R *is as in* (2.1). *Assume that* P *is projective and that it contains a unimodular element. Then the map* $\mathrm{Aut}_R(P) \to \mathrm{Aut}_A(P/XP)$ *is surjective if and only if the map* $SL_R(P) \to SL_A(P/XP)$ *is surjective.*

(2.4) *Let* $R = A[X]$ *be a polynomial ring over a local ring* A *and let* P *be a projective* R-module. *Then the map* $SL_R(P) \to SL_A(P/XP)$ *is surjective.*

Proof. See [BR].

Before we continue with further discussions it will be convenient if some of the definitions and notations are introduced.

(2.5) *Notation.* Suppose that M is an R-module and p (resp. g) is an element of M (resp. M^*). Then pg (or $p \circ g$) will denote the R-linear endomorphism of M that sends m to $g(m)p$. If p is considered as a map from R to M then this notation is consistent with that of the composition of maps.

DEFINITION. Let M, p, g be as above. If $g(p)=0$, then $1+pg$ is an automorphism of M. An automorphism of M is called a *transvection* of M if it is of the form $1+pg$ with $g(p)=0$ and if p is a unimodular element of M or if g is a unimodular element in M^*.

(2.6) *Notations.* (i) For an R-module M, $EL(M)$ will denote the subgroup of $\mathrm{Aut}_R(M)$ that is generated by all the transvections of M. If I is an ideal of R, then $EL(M, I)$ will denote the subgroup $\mathrm{Aut}_R(M)$ that is generated by transvections $1+pg$ such that $pg \equiv 0$ modulo I. If M is projective or if I is generated by a nonzero divisor, then for a generator $1+pg$ of $EL(M, I)$, as p or g is unimodular, it follows that either g is in IM^* or p is in IM.

Now follow several lemmas about these subgroups of the automorphism groups.

(2.7) LEMMA. *Suppose R is a commutative noetherian ring and I is an ideal of R. If M is an R-module, then $EL(M, I)$ is a normal subgroup of* $\mathrm{Aut}_R(M)$.

Proof. It is an immediate consequence of the fact that for a generator $1+pg$ of $EL(M)$ and an automorphism u of M, $u(1+pg)u^{-1} = 1+u(p)(gu^{-1})$.

Notation. For an R-module M, $Um(M)$ will always denote the set of all unimodular elements of M.

(2.8) LEMMA. *Let $R=A[X]$ be a polynomial ring over A. Suppose that P is a projective R-module and I is an ideal of A. If the natural map $Um(P) \to Um(P/IP)$ is surjective, then the map $EL(P, X) \to EL(P/IP, X)$ is also surjective.*

Proof. It is easy to see that the surjectivity of the map $Um(P) \to Um(P/IP)$ implies that the map $Um(P^*) \to Um((P/IP)^*)$ is also surjective.

Suppose that the transvection $1+pg$ is a generator of $EL(P/IP, X)$. That means that $pg \equiv 0$ modulo (X). Assume that p is unimodular. Then g is in $X(P/IP)^*$. So, $g = Xg'$ for some g' in $(P/IP)^*$.

Let q be in $Um(P)$ such that its image in P/IP is p. Also let f_1 and f_2 be elements in P^* such that f_1 is a lift of g' and $f_2(q)=1$. As $Xg'(p)=0$, $g'(p)=0$ and hence $f_1(q)=b$ is in IR.

Write $f = Xf_1 - bXf_2$. Then f is in XP^* and $f(q) = 0$. Clearly, $1 + qf$ is in $EL(P, X)$ and it is a lift of $1 + pg$.

In a similar way we can lift $1 + pg$ when g is unimodular. This completes the proof of (2.8).

The following is a variant of a proposition of Lindel ([L, 2.7]; see 4.3 for the statement). Like the proposition of Lindel in his paper, this proposition plays a key roll in the proof of our main theorems ((3.1) and (3.2)). We shall go into detailed discussions on Lindel's proposition [L, 2.7] in our later sections (Sects. 4, 5, 6). And now we shall state our proposition.

(2.9) PROPOSITION. *Let $R = A[X]$ be a polynomial ring over a noetherian and commutative ring A and let M be an R-module. Assume that s is in A and X is a nonzero divisor on M. Also assume that $p_1, p_2, ..., p_r$ (resp. $g_1, g_2, ..., g_r$) are elements in M (resp. M^*) such that the matrix* $(\langle g_i, p_j \rangle: i, j = 1$ *to* $r) = \text{diagonal}(1, 1, s, ..., s)$.

Write $p(X) = f_1(X)\, p_1 + f_2(X)\, p_2 + \cdots + f_r(X)\, p_r$, where $f_1, ..., f_r$ are in R such that (1) $f_1 \equiv 1$ modulo (sX), (2) *f_2 is a monic polynomial in R,* (3) *$f_i(0) = 0$ for $i = 2$ to r, and* (4) $(f_1, f_2, ..., f_r) = R$.

Then for polynomials h and h' with $h(0) = h'(0) = 0$, whenever $h - h'$ is in (sX), there are automorphisms u in $SL_2(R, sX)\, EL(M, X)$ such that $u(p(h(X))) = p(h'(X))$. (Note that substitution for X in the expression for $p(X)$ has obvious meaning.)

(2.10) *Remark.* (i) Note that under the hypothesis of (2.9) $Rp_1 + Rp_2$ can be identified with R^2. Under this identification $M = R^2 \oplus N$ where $N = \{q \text{ in } M: g_1(q) = g_2(q) = 0\}$. And that is why $SL_2(R)$ can be identified with a subgroup of $\text{Aut}_R(M)$ in a natural way. The statement of (2.9) has to be read with this natural convention in mind.

(ii) Also note that by (2.7) $EL(M, X)$ is a normal subgroup of $\text{Aut}_R(M)$ and hence $SL_2(R, sX)\, EL(M, X)$ is a subgroup of $\text{Aut}_R(M)$.

As in the proof of Lindel's proposition, we need the following lemma of Suslin.

(2.11) LEMMA [S2, Lemma 2.1]. *Let $R = A[X]$ be a polynomial ring over A and let $c = g_1(X)\, f_1(X) + g_2(X)\, f_2(X)$ be in $A \cap (f_1, f_2)$. Then for an ideal I of R and elements a and b in R, whenever $a - b$ is in* cl *there is a matrix u in $SL_2(R, I)$ such that $u(f_1(a), f_2(a)) = (f_1(b), f_2(b))$.*

Proof of (2.9). Because of (2.10), we see that $SL_2(R, sX)\, EL(M, X)$ is a subgroup of $\text{Aut}_R(M)$. We are required to prove that if h and h' are polynomials in R with $h(0) = h'(0) = 0$ and $h - h'$ in (sX), then there is an automorphism u in $SL_2(R, sX)\, EL(M, X)$ such that $u(p(h(X))) = p(h'(X))$.

Write $G = SL_2(R, sX)\, EL(M, X)$. And let $J = \{b$ in A: whenever h and h' are polynomials in R with $h(0) = h'(0) = 0$ and $h - h'$ is in (bsX), there is a u in G such that $u(p(h(X))) = p(h'(X))\}$.

It is obvious that J is an ideal of A. We shall prove that $J = A$.

First we shall prove that $A \cap (f_1, f_2)$ *is contained in* J.

To prove this let $b = d_1 f_1 + d_2 f_2$ be in A and let h and h' be in R such that $h(0) = h'(0) = 0$ and $h - h'$ is in (bsX). Then by Suslin's lemma (2.10), there is an automorphism u in $SL_2(R, sX)$ such that $u(f_1(h(X)), f_2(h(X))) = (f_1(h'(X)), f_2(h'(X)))$.

As $h - h'$ is in (bsX), $f(h(X)) - f(h'(X))$ is also in (bsX) for all polynomials f in R. Therefore, $p(h(X)) = f_1(h(X))\, p_1 + f_2(h(X))\, p_2 + f_3(h'(X))\, p_3 + \cdots + f_r(h'(X))\, p_r - bw$ for some w in sXN.

Write $u_1 = (1 + d_1(h(X))wg_1)(1 + d_2(h(X))wg_2)$. Then u_1 is in $EL(M, X)$ and $u_1(p(h(X))) = p(h(X)) + d_1(h(X))\, f_1(h(X))w + d_2(h(X))\, f_2(h(X))w = p(h(X)) + bw = f_1(h(X))\, p_1 + f_2(h(X))\, p_2 + f_3(h'(X))\, p_3 + \cdots + f_r(h'(X))\, p_r$. And hence $uu_1(p(h(X)) = p(h'(X))$. This shows that b in in J. That means that $A \cap (f_1, f_2)$ is contained in J, as it was claimed.

To prove that $J = A$, we assume the contrary that J is contained in a maximal ideal $\mathbf{m}$ of A. Since $(f_1, f_2) + sR = R$ and f_2 is monic, it follows that $(f_1, f_2) \cap A + sA = A$. As $(f_1, f_2) \cap A$ is contained in J, s does not belong to $\mathbf{m}$.

Again as $f_1 \equiv 1$ modulo (sX), there are polynomials $c_3, \ldots, c_r$ in R such that $(c, f_2) + \mathbf{m}R = R$, where $c = f_1 + sXc_3 f_3 + \cdots + sXc_r f_r$.

Now we shall prove that $(c, f_2) \cap A$ is contained in J and hence in $\mathbf{m}$.

Suppose that b is in $(c, f_2) \cap A$ and h and h' are polynomials in R with $h(0) = h'(0) = 0$ and $h - h'$ is in (sbX).

Let $u_1 = (1 + p_1 h(X)\, c_3(h(X))\, g_3)(1 + p_1 h(X)\, c_4(h(X))\, g_4) \cdots (1 + p_1 h(X)\, c_r(h(X))\, g_r)$. Then u_1 is in $EL(M, X)$ and $u_1(p(h(X))) = c(h(X))\, p_1 + f_2(h(X))\, p_2 + \cdots + f_r(h(X))\, p_r$.

In a similar way there is u_2 in $EL(M, X)$ such that $u_2(p(h'(X))) = c(h'(X))\, p_1 + f_2(h'(X))\, p_2 + \cdots + f_r(h'(X))\, p_r$.

By Suslin's lemma (2.11), we can find an automorphism u_3 in $SL_2(R, sX)$ such that $u_3(c(h(X))\, p_1 + f_2(h(X))\, p_2 + f_3(h(X))\, p_3 + \cdots + f_r(h(X))\, p_r) = c(h'(X))\, p_1 + f_2(h'(X))\, p_2 + f_3(h(X))\, p_3 + \cdots + f_r(h(X))\, p_r$. As in the previous claim, we can also see that there is u_4 in $EL(M, X)$ such that $u_4(c(h'(X))\, p_1 + f_2(h'(X))\, p_2 + f_3(h(X))\, p_3 + \cdots + f_r(h(X))\, p_r) = c(h'(X))\, p_1 + f_2(h'(X))\, p_2 + f_3(h'(X))\, p_3 + \cdots + f_r(h'(X))\, p_r$.

So, if we write $u = u_2^{-1} u_4 u_3 u_1$, then u is in $SL_2(R, sX)\, EL(M, X)$ and $u(p(h(X))) = p(h'(X))$. Hence b is in J. So, as we have claimed, it is established that $(c, f_2) \cap A$ is contained in J and hence in $\mathbf{m}$.

But this leads to a contradiction because we already have that $(c, f_2) + \mathbf{m}R = R$ and that f_2 is monic and hence $(c, f_2) \cap A + \mathbf{m} = A$. This shows that $J = A$. Therefore Proposition (2.9) is established.

3. On Lifting of Automorphisms

In this section we shall prove our main results on the lifting problem.

(3.1) Theorem. *Let A be a commutative noetherian ring of finite Krull dimension and let $R = A[X_1, ..., X_n]$ be a polynomial ring over A. Suppose that P is a projective R-module with* $\operatorname{rank}(P_y) > \dim A$ *for all y in* $\operatorname{Spec} R$. *Then the natural map* $\operatorname{Aut}_R(P) \to \operatorname{Aut}_{A'}(P/X_nP)$ *is surjective, where* $A' = A[X_1, ..., X_{n-1}]$.

Proof. By standard arguments we can assume that A is reduced and that P has constant rank. Because of (2.2), we can also assume $\operatorname{rank}(P) > 1$.

Now let g be an automorphism of P/X_nP. We would like to lift g to an automorphism of P.

By "barring" we shall always denote "modulo X_n."

We can find a projective R-module Q such that $F = P \oplus Q$ is free. Obviously, $g \oplus \mathrm{Id}_{\bar{Q}}$ can always be lifted and hence $g \oplus \mathrm{Id}_{\bar{Q}} \oplus \mathrm{Id}_{\bar{P}}$ can also be lifted to an automorphism. This means that there is an automorphism $H: P \oplus F \to P \oplus F$ such that $\bar{H} = g \oplus \mathrm{Id}_{\bar{F}}$.

By using downward induction on the rank of the free module F, it is enough to prove that *if $g: \bar{P} \to \bar{P}$ is an automorphism of P and if $H: P \oplus R \to P \oplus R$ is also an automorphism of $P \oplus R$ such that $\bar{H} = g \oplus \mathrm{Id}_{\bar{R}}$, then there is an automorphism $G: P \to P$ of P such that $\bar{G} = g$.*

Now (3.1) will follow from the following theorem (3.2).

(3.2) Theorem. *Let A be a commutative noetherian ring with finite Krull dimension and let $R = A[X_1, ..., X_n]$ be a polynomial ring over A. Suppose that P is a projective R-module with constant rank strictly bigger than* $\max(1, \dim A)$. *Then, if (a, p) is a unimodular element in $R \oplus P$ with* $(a, p) \equiv (1, 0)$ *modulo X_n, then there is an automorphism u in $EL(R \oplus P, X_n)$ such that $u(a, p) = (1, 0)$.*

Proof. Since $\operatorname{rank} P > \max(1, \dim A)$, by [BR], $P = R \oplus P'$ for some projective R-module with $\operatorname{rank} P' \geqslant 1$. Write $Q = R \oplus P = R^2 \oplus P$ and $r = \operatorname{rank} Q > \max(2, \dim A + 1)$.

We shall use induction on $\dim A$ to prove the theorem. Also note that at each step of the induction it is enough to assume that A is reduced.

If $\dim A = 0$, then A, being reduced, is a product of fields. Therefore, by the theorem of Quillen [Q] and Suslin [S1] P' is free. As $(a, p) \equiv (1, 0)$ modulo X_n, by Suslin's theorem [S2, Corollary 2.5] it follows that there is u in $EL(Q, X_n)$ such that $u(a, p) = (1, 0)$. So the theorem is proved for $\dim A = 0$.

Now we shall assume that $\dim A > 0$.

Let S be the set of all nonzero divisors of A. Then $S^{-1}P'$ is free [Q, S1] of rank $r-2$. So, we can find a nonzero divisor s of A and a free submodule F of P' with basis $p_3, ..., p_r$ such that $F_s = P_s$. By replacing s by an appropriate power of s, we may assume that there are elements $g_3, ..., g_r$ in P'^* such that $(\langle g_i, p_j \rangle : 3 \leqslant i, j \leqslant r) = \text{diagonal}(s, ..., s)$ and such that sP' is contained in F.

Let p_1 and p_2 denote, respectively, the elements $(1, 0, 0)$ and $(0, 1, 0)$ in $Q = R^2 \oplus P$. And for $i = 3$ to r, we can extend g_i to elements in Q^* by putting $g_i(p_1) = g_i(p_2) = 0$. We define g_1 and g_2 in Q^* such that $(\langle g_i, p_j \rangle : 1 \leqslant i, j \leqslant 2) = \text{diagonal}(1, 1)$ and $g_{i|P'} = 0$. Thus we have the matrix $(\langle g_i, p_j \rangle : 1 \leqslant i, j \leqslant r) = \text{diagonal}(1, 1, s, ..., s)$.

We shall write $X_n = X$ and $a = f_1$. We can write $(a, p) = f_1 p_1 + f_2 p_2 + p'$, where f_1 and f_2 are polynomials in R and p' is in P'. As $(a, p) \equiv (1, 0)$ modulo (X), $f_1(0) = 1$, $f_2(0) = 0$, and p' is in XP'.

Since $\dim(A/sA) < \dim A$, by the induction hypothesis there is u' in $EL(Q/sQ, X)$ such that $u'(f_1 q_1 + f_2 q_2 + q') = q_1$, where q_1, q_2, q' are, respectively, the images of p_1, p_2, p' in Q/sQ. By [L, Proposition 1.12], the natural map $Um(Q) \to Um(Q/sQ)$ is surjective. Hence by (2.8), there is u in $EL(Q, X)$ that lifts u'. Hence by replacing $f_1 p_1 + f_2 p_2 + p'$ by its image under u, we can assume that $f_1 \equiv 1$ modulo (sX), f_2 is in (sX), and p' is in sXP'.

Since $f_1 p_1 + f_2 p_2 + p'$ is in $Um(Q)$ and $f_1 \equiv 1$ modulo(sX), $f_1 p_1 + sXf_2 p_2 + p'$ is also in $Um(Q)$. Therefore, by the theorem of Eisenbud and Evans [EE], there are elements h_1 in R and p'' in P' such that ideal $R(f_1 + sXh_1 f_2) + O(p' + sXf_2 p'')$ has height at least $\dim A + 1$. Therefore, after a change of variables that sends X_i to $X_i + X^N$ for $i = 1$ to $n-1$ and X to X, where N is large enough, we can assume that the ideal $R(f_1 + sXh_1 f_2) + O(p' + sXf_2 p'')$ contains a monic polynomial $h(X)$ with coefficients in $A[X_1, ..., X_{n-1}]$.

We shall write $B = A[X_1, ..., X_{n-1}]$ and $R = B[X]$. We also write $h(X) = h'(X)(f_1 + sXh_1 f_2) + g(p' + sXf_2 p'')$ for some g in P'^*. Let d be a positive integer such that $f_2 + X^d h(X)$ is a monic polynomial in X with coefficients in B. We shall again regard g as an element of Q^* by putting $g(p_1) = g(p_2) = 0$.

We shall define elements u_i for $i = 1$ to 4 in $EL(Q, X)$ as follows.

Put $u_1 = 1 + p_1 sXh_1 g_2$, $u_2 = 1 + sXp'' g_2$, $u_3 = 1 + p_2 X^d h'(X) g_1$, $u_4 = 1 + p_2 X^d g$. Then by replacing $f_1 p_1 + f_2 p_2 + p'$ by $u_4 u_3 u_2 u_1 (f_1 p_1 + f_2 p_2 + p')$ we may assume that $f_1 \equiv 1$ modulo(sX), f_2 is a monic polynomial with coefficients in B, and $f_2(0) = 0$ and also that p' is in sXP'. Since sP' is contained in F, we have that $p' = X(f_3 p_3 + f_4 p_4 + \cdots + f_r p_r)$ for some polynomials $f_3, ..., f_r$ in R.

Thus $f_1 p_1 + f_2 p_2 + p' = f_1 p_1 + f_2 p_2 + Xf_3 p_3 + \cdots + Xf_r p_r$. Since $f_1 \equiv 1$ modulo(sX) and f_2 is monic, there is b in B such that $1 - sb$ is in (f_1, f_2).

Moreover, since $f_1 p_1 + f_2 p_2 + Xf_3 p_3 + \cdots + Xf_r p_r$ is in $Um(Q)$ and Q_s is free with basis $p_1, ..., p_r$ and $f_1 \equiv 1$ modulo(sX), it follows that $(f_1, f_2, Xf_3, ..., Xf_r) = R$.

By an application of (2.9) with $h(X) = X$ and $h'(X) = (1 - sb)X$, we get automorphisms u_5 in $EL(Q, X)$ and u_6 in $SL_2(R, sX)$ such that $u_5(f_1 p_1 + f_2 p_2 + Xf_3 p_3 + \cdots + Xf_r p_r) = f'_1 p_1 + f'_2 p_2 + (1 - sb) Xf_3((1 - sb)X) p_3 + \cdots + (1 - sb) Xf_r((1 - sb)X) p_r$, where $u_6(f'_1, f'_2) = (f_1((1 - sb)X), f_2((1 - sb)X))$.

Note that $f_i(X) - f_i((1 - sb)X)$ is in (sX) for $i = 1$, 2 and also as u_6 is in $SL_2(R, sX)$, $f_i((1 - sb)X) - f'_i(X)$ is in (sX) for $i = 1,2$. Hence $f'_1 \equiv f_1 \equiv 1$ modulo(sX) and $f'_2(0) = f_2(0) = 0$. It also follows that $1 - sb$ is in $(f'_1, f'_2) \cap B = (f_1((1 - sb)X), f_2((1 - sb)X)) \cap B$.

As $f'_1 p_1 + f'_2 p_2 + (1 - sb) Xf_3((1 - sb)X) p_3 + \cdots + (1 - sb) Xf_r((1 - sb)X) p_r$ is unimodular in Q and $1 - sb$ is in (f'_1, f'_2), it follows that $f'_1 p_1 + f'_2 p_2$ is unimodular in Q. Therefore, there is g in Q^* such that $g(f'_1 p_1 + f'_2 p_2) = 1$ and $g_{|P'} = 0$.

Let $v_3 = 1 + (X + (sb - 1) Xf_3((1 - sb)X)) p_3 g$ and $v_i = 1 + (sb - 1) Xf_i((1 - sb)X) p_i$ for $i = 4$ to r. If $v = v_3 v_4 \cdots v_r$ then v is in $EL(Q, X)$ and $vu_5(f_1 p_1 + f_2 p_2 + Xf_3 p_3 + \cdots + Xf_r p_r) = v(f'_1 p_1 + f'_2 p_2 + (1 - sb) Xf_3((1 - sb)X) p_3 + \cdots + (1 - sb) Xf_r((1 - sb)X) p_r) = f'_1 p_1 + f'_2 p_2 + Xp_3$.

Now we can write $f'_1 = 1 + sXf''_1$ and $f'_2 = Xf''_2$ for some polynomials f''_1 and f''_2 in R. Let $U_1 = 1 - p_1 f''_1 g_3$ and $U_2 = 1 - p_2 Xf''_2 g_1$ and $U_3 = 1 - Xp_3 g_1$ and put $U = U_1^{-1} U_3 U_2 U_1$. Then since U_2 and U_3 are in $EL(Q, X)$, U is also in $EL(Q, X)$. Finally, $U(f'_1 p_1 + f'_2 p_2 + Xp_3) = p_1$.

This shows that there is an automorphism u in $EL(Q, X)$ such that $u(a, p) = (1, 0)$. Therefore the proof of (3.2) is complete.

(3.3) COROLLARY. *Suppose that* $R = A[X_1, ..., X_n]$ *is a polynomial ring over a commutative noetherian ring* A *and* P *is a projective* R*-module with* $\operatorname{rank}(P_y) > \dim A$ *at all* y *in* $\operatorname{Spec} R$. *Then the natural map* $\operatorname{Aut}_R(P) \to \operatorname{Aut}_{R'}(P/(X_{r+1}, ..., X_n)P)$ *is surjective, where* $0 \leqslant r \leqslant n$ *and* $R' = A[X_1, ..., X_r]$.

In particular, if $R = A[X]$ *and* P *is a projective* R*-module with* $\operatorname{rank} P_y > \dim A$ *at all* y *in* $\operatorname{Spec} R$ *then the map* $\operatorname{Aut}_R(P) \to \operatorname{Aut}_A(P/XP)$ *is surjective.*

In view of all these discussions, we pose the following problem.

(3.4) *Conjecture.* Suppose $R = A[X]$ is a polynomial ring over A and P is a finitely generated projective R-module. Then, is the natural map $\operatorname{Aut}_R(P) \to \operatorname{Aut}_A(P/XP)$ surjective?

(3.5) *Remark.* We have already made some comments about (3.4) in our preliminary discussions (Sect. 2). It is also clear that to prove (3.4) *we* *can assume that A is reduced and P has constant rank.* We shall also give an example (6.3) to show that (3.4) fails when P is not projective.

With this we close our discussions on lifting of automorphisms.

4. Action of $SL_2(R)\, EL(R^2 \oplus M)$ on Special Unimodular Elements

A careful analysis of the work of Lindel (see [L, 2.7] and our proof of the lifting theorem (see (2.9)) reveals that, for a finitely generated module M over a polynomial ring $R = A[X]$, the study of the action of $SL_2(R)\, EL(R^2 \oplus M)$ on the set of special (type of) unimodular elements of $R^2 \oplus M$ plays an important roll in both the cases.

As a technical tool, this kind of study was initiated by Lindel [L, 2.7]. In the rest of this paper we shall be studying when this kind of an action is transitive on the set of all special (type of) unimodular elements of $R^2 \oplus M$.

In this section we set up some more notations, preliminaries that we need for the rest of our discussions and we also make some remarks on the work of Lindel [L]. In fact, the work of Lindel [L] is the main motivation for the discussions in this section and later. In Sections 5 and 6 we shall, respectively, discuss our results and a counterexample related to this investigation.

Preliminaries and a Remark on Lindel's Work

As before R and A will always denote noetherian and commutative rings. We shall first define *special unimodular elements* and fix some notations.

(4.1) Definition. Let $R = A[X]$ be a polynomial ring over A and let M be an R-module. A unimodular element $(f_1(X), f_2(X), m)$ of $R^2 \oplus M$ will be called a *special unimodular element* if f_1 is a monic polynomial.

(4.2) *Notations.* As in (2.10), note that $SL_2(R)$ is a subgroup of $\mathrm{Aut}_R(R^2 \oplus M)$ in a natural way. And as before, by $SL_2(R)\, EL(R^2 \oplus M)$ we shall mean the subgroup of $\mathrm{Aut}_R(R^2 \oplus M)$ generated by $SL_2(R)$ and $EL(R^2 \oplus M)$.

Now we are ready to discuss the work of Lindel.

In [L], Lindel proved that [L, 2.8] *for projective modules P over $R = A[X_1, ..., X_n]$ with* rank $P > \dim A$, *$EL(R \oplus P)$ acts transitively on the set of all unimodular elements of $R \oplus P$.* In his proof he used a very elegant proposition, the variant (2.9) of which was of much use to us. We shall state Lindel's proposition.

(4.3) PROPOSITION [L, (2.7)]. *Let $R = A[X]$ be a polynomial ring over A and let M be an R-module. Suppose $p_3, ..., p_m$ (resp. $g_3, ..., g_m$) are elements in M (resp. $\mathrm{Hom}(M, R)$) such that $(g_i(p_j))_{3 \leqslant u, j \leqslant m} = \mathrm{diag}(s_3, ..., s_m)$ for some $s_3, ..., s_m$ in A. If $(f_1, f_2, s_3 f_3, ..., s_m f_m)$ is a unimodular row in R^m with f_1 monic, then for all a and b in R there is an automorphism U in $SL_2(R)\, EL(R^2 \oplus M)$ such that $U(f_1(a), f_2(a), f_3(a)\, p_3 + \cdots + f_m(a)\, p_m) = (f_1(b), f_2(b), f_3(b)\, p_3 + \cdots + f_m(b)\, p_m)$.*

This elegant proposition can also be improved a little further. Although this improvement will still remain technical, it may be worthwhile to mention it.

(4.4) PROPOSITION. *Suppose we have a situation as in* (4.3). *Then for all b in R, there is an automorphism U in $SL_2(R)\, EL(R^2 \oplus M)$ such that $U(f_1(b), f_2(b), f_3(b)\, p_3 + \cdots + f_m(b)\, p_m) = (1, 0, 0)$.*

Proof. By (4.3) it is enough to assume $b = X$.

We have $f_1(X)$ is a monic polynomial and $(f_1(X), f_2(X), s_3 f_3(X), ..., s_m f_m(X))$ is a unimodular row. Let T be an indeterminate over R. Then for suitable integers $r_l, ..., r_m$ if $F_1(T) = T^r 1 f_1(X - T + T^{-1})$, $F_2(T) = T^r 2 f_2(X - T + T^{-1}), ..., F_m(T) = T^r m f_m(X - T + T^{-1})$, then F_1 is a monic polynomial in T and $F_1(0) = 1$ and $F_i(T)$ is in $TR[T]$ for $i = 2$ to m. Since the A-algebra map $R[T, T^{-1}] \to R[T, T^{-1}]$ that sends X to $X - T + T^{-1}$ and T to T is an automorphism and $F_1(0) = 1$, it follows that $(F_1(T), F_2(T), s_3 F_3(T), ..., s_m F_m(T))$ is a unimodular row in $R[T]^m$.

By (4.3), there is an automorphism u in $SL_2(R[T])\ EL(R[T]^2 \oplus M \otimes R[T])$ such that $u(F_1(T), F_2(T), F_3(T)\, p_3 + \cdots + F_m(T)\, p_m) = (F_1(0), F_2(0), F_3(0)\, p_3 + \cdots + F_m(0)\, p_m) = (1, 0, 0)$.

Now by "substituting $T = 1$" (i.e., by tensoring with $R[T]/(T-1)\, R[T]$) we get an automorphism $v = u$ (modulo $T - 1$) in $SL_2(R)\ EL(R^2 \oplus M)$ such that $v(f_1(X), f_2(X), f_3(X)\, p_3 + \cdots + f_m(X)\, p_m) = (1, 0, 0)$. This completes the proof of (3.4).

The idea of the proofs of both (4.3) and (4.4) will be used in what follows in Section 5.

5. TRANSITIVITY OF $SL_2(R)\, EL(R^2 \oplus M)$ WHEN M IS EXTENDED

With (4.3) and (4.4) in mind, for an $R = A[X]$-module M, it may be natural to ask, under what conditions on M does $SL_2(R)\, EL(R^2 \oplus M)$ act transitively on the set of all special unimodular elements of $R^2 \oplus M$. If we assume that M is extended from A, then (5.2) settles this question. We shall also see (6.2) that we cannot expect such a statement to hold always.

Our key result (5.1) extends the result of Suslin [S2], Proposition (3.4)] where M was assumed to be free. Before we state (5.1), we shall fix some *notations and conventions.*

For a polynomial ring $R=A[X]$ over a commutative ring A and an extended R-module $M=M_0\otimes R$, where M_0 is an A-module, an element p of M can be thought of as a "*polynomial*" with coefficients in M_0. For this reason, an element p of M will often be denoted by $p(X)$ and substitution for X will make perfect sense. Similarly, elements of $\mathrm{Hom}_R(M, R)$ can also be thought of as polynomials with coefficients in $\mathrm{Hom}_A(M_0, A)$.

(5.1) THEOREM. *Let $R=A[X]$ be a polynomial ring over A and let M be an extended R-module. Suppose that $p(X)=(f_1(X), f_2(X), p_3(X))$ is a special unimodular element in $R^2\oplus M$ with f_1 monic. Also suppose that I is an ideal of R and write $G=\{u$ in $SL_2(R)\,EL(R^2\oplus M)$: $u\equiv \mathrm{Id}$ modulo $I\}$. Then for elements a, b in R, whenever $a-b$ is in I, there is an automorphism u in G such that $u(p(a))=p(b)$.*

Proof. Let $J=\{c$ in A: for a, b in R, whenever $a-b$ is in cl, there is u in G such that $u(p(a))=p(b)\}$. It is a routine checking that J is an ideal. We only need to show that $J=A$.

Since $p(X)$ is unimodular in $R^2\oplus M$, there is an element $h_3(X)$ in $\mathrm{Hom}_R(M, R)$ such that $Rf_1+Rf_2+R\langle h_3(X), p_3(X)\rangle=R$.

We claim that if $g_2(X)=f_2(X)+\langle h_3(X), p_3(X)\rangle\, d(X)$ for some polynomial $d(X)$, then $(Rf_1+Rg_2)\cap A$ is contained in J.

Suppose $c=r_1(X)\,f_1(X)+r_2(X)\,g_2(X)$ is in A and $a-b$ is in cl. Then we are required to show that there is u in G such that $u(p(a))=p(b)$.

Write $p_1=(1,0,0)$ and $p_2=(0,1,0)$. Define $u_1=1+p_2\,d(a)\,h_3(a)$ and $u_2=1+p_2\,d(b)\,h_3(b)$, where we consider $d(a)\,h_3(a)$ and $d(b)\,h_3(b)$ as elements in $(R^2\oplus M)^*$. Then u_1 and u_2 are in $EL(R^2\oplus M)$. Moreover $u_1(p(a))=(f_1(a), g_2(a), p_3(a))=q_1$ (say) and $u_2(p(b))=(f_1(b), g_2(b), p_3(b))$ $=q_2$ (say).

Write $q_3=(f_1(a), g_2(a), p_3(b))$; then $q_1-q_3=(0,0,p_3(a)-p_3(b))=cw$ for some w in IM.

Let pr_1 and pr_2 be, respectively, the projections from $R^2\oplus M$ to the first and the second coordinates. Define $u_3=(1-r_1(a)wpr_1)(1-r_2(a)wpr_2)$. Then, as w is in IM, u_3 is in $EL(R^2\oplus M, I)$. Also $u_3(q_1)=(f_1(a), g_2(a), p_3(a)-cw))=q_3$.

Since $c=r_1(X)\,f_1(X)+r_2(X)\,g_2(X)$ is in $(Rf_1+Rf_2)\cap A$, by Suslin's Lemma (2.11) there is u_4 in $SL_2(R, I)$ such that $u_4(f_1(a), g_2(a))=(f_1(b), g_2(b))$ and hence $u_4(q_3)=(f_1(b), g_2(b), p_3(b))$.

Hence it follows that if $u=u_2^{-1}u_4u_3u_1$, then u is in $SL_2(R)\,EL(R^2\oplus M)$ and $u(p(a))=p(b)$. In fact, as $u_1\equiv u_2$ modulo I, u is in G. This proves that c is in J. Hence, as it was claimed, $(Rf_1+Rg_2)\cap A$ is contained in J, whenever g_2 is of the form $f_2(X)+\langle h_3(X), p_3(X)\rangle\, d(X)$.

Finally, we prove that $J = A$. This part of the proof is also as that in Lindel's proof of (4.3) or that of (2.9).

Suppose J is contained in a maximal ideal $\mathbf{m}$ of A. Then $R/\mathbf{m}R + f_1R$ is a semilocal ring of dimension zero. The image of $(f_2(X), \langle h_3(X), p_3(X)\rangle)$ in $(R/\mathbf{m}R + f_1R)^2$ is a unimodular row. Hence it follows that there is a polynomial $d(X)$ in R such that the image of $f_2(X) + \langle h_3(X), p_3(X)\rangle\, d(X)$ in $R/\mathbf{m}R + f_1R$ is a unit. Hence, if $g_2(X) = f_2(X) + \langle h_3(X), p_3(X)\rangle$ $d(X)$, then $Rf_1 + Rg_2 + \mathbf{m}R = R$. Since f_1 is monic, it follows that $(Rf_1 + Rg_2) \cap A + \mathbf{m} = A$. Since we have seen that $(Rf_1 + Rg_2) \cap A$ is contained in J, $J + \mathbf{m} = A$. This contradicts that J is contained in $\mathbf{m}$. Therefore the proof of (5.1) is complete.

Our main result (5.2) in this section is a consequence of (5.1). There are other consequences listed below.

(5.2) THEOREM. *Suppose R and M are as in* (5.1) *and* $p(X) = (f_1(X), f_2(X), p_3(X))$ *is a special unimodular element in* $R^2 \oplus M$ *with* f_1 *monic. Then for all a in R there is an automorphism u in* $SL_2(R)$ $EL(R^2 \oplus M)$ *such that* $u(p(a)) = (1, 0, 0)$. *In particular,* $SL_2(R)$ $EL(R^2 \oplus M)$ *acts transitively on the set of all special unimodular elements of* $R^2 \oplus M$.

Proof. By (5.1) we can assume that $a = X$.

Let T be an indeterminate over R and let $F_1(T) = T^d f_1(X - T + T^{-1})$ where $d = \text{degree}(f_1)$. Again, for some suitable integer r, $F_2(T) = T^r f_2(X - T + T^{-1})$ is in $TR[T]$ and $q_3(T) = T^r p_3(X - T + T^{-1})$ is in $T(M \otimes R[T])$.

As $p(X)$ is unimodular in $R[T, T^{-1}]^2 \oplus M \otimes R[T, T^{-1}]$, it follows that $p(X - T + T^{-1})$ is also unimodular. Hence $q(T) = (F_1(T), F_2(T), q_3(T))$ is also unimodular in $R[T, T^{-1}]^2 \oplus M \otimes R[T, T^{-1}]$. As $F(0) = 1$, it follows that $q(T)$ is a special unimodular element in $R[T]^2 \oplus M \otimes R[T]$.

So, by (5.1), there is an automorphism U in $SL_2(R[T])$ $EL(R[T]^2 \oplus M \otimes R[T])$ such that $U(q(T)) = q(0) = (1, 0, 0)$. By "substituting $T = 1$" we get an automorphism u in $SL_2(R)\, EL(R^2 \oplus M)$ such that $u(q(1)) = (1, 0, 0)$; i.e., $u(p(X)) = (1, 0, 0)$. This completes the proof of (5.2).

Using another standard argument of Suslin, we have the following corollary.

(5.3) COROLLARY. *Suppose R and M are as in* (5.1). *Then* $EL(R^3 \oplus M)$ *acts transitively on the set of all special unimodular elements of* $R^3 \oplus M$.

Remark. When M is free, (5.3) was proved by Rao [R].

The following cancellation theorem of Swan [Sw] also follows from (5.1).

(5.4) COROLLARY (Swan). *Suppose* $R = A[X_1, ..., X_n]$ *is a polynomial ring over* A *and* P *is a finitely generated projective* R*-module that is extended from* A. *If* $\operatorname{rank}(P) > \dim A$, *then* P *is cancellative.*

Proof. Suppose $f: Q \oplus R \to P \oplus R$ is an isomorphism. We have to show that P is isomorphic to Q.

Let $f(0, 1) = (p, a)$. By standard arguments, we can assume that a is a monic polynomial in X_n. Also note that, since P is extended from A with rank $P > \dim A$, by Serre's theorem $P = M \oplus R$ for some extended projective R-module M. Now it follows that (p, a) is a special unimodular element in $M \oplus R^2$ and hence by (5.2), there is u in $SL_2(R)\,EL(P \oplus R)$ such that $u(p, a) = (0, 1)$. Hence (5.4) is established.

6. A QUESTION AND COUNTEREXAMPLES

In view of (4.4) and (5.2), the following is a very natural question.

(6.1) *Question.* Suppose $R = A[X]$ is a polynomial ring over A and P is a finitely generated projective R-module. Does $SL_2(R)\,EL(R^2 \oplus P)$ act transitively on special unimodular elements of $R^2 \oplus P$?

For projective modules of large enough rank, Lindel gave an affirmative answer (see [L, (2.8)]) to this question (6.1).

We also give a counterexample (6.2) that if P is nonprojective then (6.1) does not have an affirmative answer.

(6.2) EXAMPLE. Let $S = K[X_1, X_2, X_3]/(X_1^2 + X_2^2 + X_3^2 - 1)$ be the coordinate ring of a sphere over the field, K of real numbers. Suppose P is the kernel of the map $S^3 \to S$ that sends the standard basis e_1, e_2, e_3 to the image of X_1, X_2, X_3 in S. Clearly $P \oplus S$ is isomorphic to S^3, but it is well known that P is not isomorphic to S^2.

Eisenbud and Evans [EE, Example] used this example to produce some interesting counterexamples. We are going to use their example for our purpose.

Write $R = S[Y_1, Y_2]$ and $I = (Y_1, Y_2)R$, $Q = P \otimes R$, $M = Q \oplus I$ and $N = R^2 \oplus I$. Clearly, $R \oplus M = R \oplus N$. It was shown in [EE, 8, Example] that M is not isomorphic to N.

Suppose $U: R \oplus M \to R \oplus N$ is an isomorphism and $U(1, 0) = (f_1, f_2, f_3, f_4)$ in $R \oplus N = R^3 \oplus I$.

Since $\operatorname{Hom}(I, R) = R$ (see the proof of [EE, 8, Lemma 7]), it follows that (f_1, f_2, f_3, f_4) is a unimodular row in R^4. Hence by [EE, 3, Corollary 2] there is g_1, g_2, g_3 in R such that $F_1 = f_1 + g_1 f_4$, $F_2 = f_2 + g_2 f_4$, $F_3 = f_3 + g_3 f_4$ then $\operatorname{height}(RF_1 + RF_2 + RF_3) \geqslant 3$.

As dim $S=2$, we can make a change of variables $Y_1 \to Y_1 + Y_2^r$, $Y_2 \to Y_2$ and assume that $H_1F_1+H_2F_2+H_3F_3=F$ is monic in Y_2, for some H_1, H_2, H_3. We can assume that H_1, H_2, H_3 is in Y_2R, and hence in I, and also that Y_2 − degree$(F) > Y_2$ − degree(f_4).

Define u_1, u_2, u_3 in $EL(R^3 \oplus I)$ by $u_1(a_1, a_2, a_3, a_4) = (a_1 + g_1a_4, a_2+g_2a_4, a_3+g_3a_4, a_4)$, $u_2(a_1, a_2, a_3, a_4) = (a_1, a_2, a_3, a_4 + a_1H_1 + a_2H_2 + a_3H_3)$, and $u_3(a_1, a_2, a_3, a_4) = (a_1 + a_4Y_2^t, a_2, a_3, a_4)$ for all (a_1, a_2, a_3, a_4) in $R^3 \oplus I$.

For high enough t, if $h = u_3u_2u_1$ then $h(f_1, f_2, f_3, f_4)$ is a special unimodular element of $R^3 \oplus I$. And there is no automorphism of $R^3 \oplus I$ that sends $h(f_1, f_2, f_3, f_4)$ to $(1, 0)$, because otherwise M will be isomorphic to N.

This shows that $SL_2(R)\, EL(R^3 \oplus I)$ does not act transitively on the set of special unimodular elements.

The following example will show that Conjecture (3.4) fails for non-projective modules.

(6.3) EXAMPLE. Suppose $(A, \mathbf{m})$ is a discrete valuation ring and $R = A[X]$. Let $M = (\mathbf{m}, X)$. Then M/XM is isomorphic to $\mathbf{m} \oplus A/\mathbf{m}$. As in (6.2), note that $\mathrm{Hom}_R(M, R) = R$ and hence $\mathrm{Hom}_R(M, M) = R$ and $\mathrm{Aut}_R(M)$ = units of R. As $\mathbf{m} \oplus A/\mathbf{m}$ has automorphisms that are not multiplication by units, it follows that the map $\mathrm{Aut}_R(M) \to \mathrm{Aut}_A(m/XM)$ is not surjective.

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