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Ideals as Sections of Projective Modules

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1. Introduction

Let A be a reduced affine algebra of dimension n over an algebraically closed field k. Assume that F^nK_0A has no $(n-1)!$ torsion. Let P be a projective A-module of rank n and I be an ideal of A. In [5], it was shown that there is a surjective map $f : P \twoheadrightarrow A$ if and only if the n-th Chern class $C_n(P) = 0$ (see [5]). We consider here the obstruction for the existence of a surjective map $f : P \twoheadrightarrow I$. In fact, we here start with a surjective map $\bar{f} : P \twoheadrightarrow I/I^2$ and try to find precise obstruction for "extending" $\bar{f}$ to a surjective map $f : P \twoheadrightarrow I$. In case I is a local complete intersection ideal of height n, we show (see Theorem 2.1) that such an extension $f : P \to I$ exists if and only if $C_n(P^*) =$ cycle associated to (A/I), where $P^* = \mathrm{Hom}(P, A)$. For example (see Corollary 2.2), it follows immediately from this result (Theorem 2.1) that if I is a complete intersection ideal of height n, then any set of n-generators of I/I^2 lifts to a set of n-generators of I.

We also give examples (see Example 2.3, Example 2.4) that this result (Corollary 2.2) fails, when k is not algebraically closed or when I is not a local complete intersection of height n.

When I is any ideal of A, we give precise obstruction for existence of a surjective map from $P \twoheadrightarrow I$, in terms of Segre class of I (see Theorem 2.6). The proofs of Theorem 2.1 and Theorem 2.6 are based on the methods of [5].

In Section 3, we consider the case $k = \bar{\mathbb{F}}_q$ and show that any surjective map $\bar{f} : P \twoheadrightarrow I/I^2$ extends to a surjective map $f : P \twoheadrightarrow I$. Here the proof uses some of the techniques from [10].

We fix some notations and terminology before we close this section. For a reduced affine algebra A of dimension n over an algebraically closed field, we denote by K_0A the Grothendieck group of projective A-modules, which is in fact, equal to the Grothendieck group of finitely generated A-modules of finite projective dimension. We denote by F^nK_0A, the subgroup of K_0A, generated by the classes (A/m), where m is a smooth maximal ideal of height n. For a projective A-module P of rank n we define the n-th Chern class of $P = C_n(P) = (-1)^n \sum_{i=0}^{n}(-1)^i[\Lambda^i P^*]$. For intersection theory used here, we refer to [4]. For other unexplained definitions and notations, we refer to [5].

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2. Affine Algebras over Algebraically Closed Fields

Theorem 2.1. *Let A be a reduced affine algebra of dimension $n \geq 2$ over an algebraically close field k. Let P be a projective A-module of rank n and I be a local complete intersection ideal of height n in A. Let $\bar{f} : P/IP \twoheadrightarrow I/I^2$ be a surjective map. Suppose that F^nK_0A has no $(n-1)!$ torsion. Then there exists a surjective map $f : P \to I$ such that $f \otimes A/I = \bar{f}$ if and only if $C_n(P^*) = (A/I)$.*

Proof. It is clear that if there is a surjective map $f : P \twoheadrightarrow I$ then $C_n(P^*) = (A/I)$. So, we need only to prove if part of the theorem.

By Bertini's theorem [11, 5], choose a "general lift" $g : P \to I$ of $\bar{f}$ such that $g \otimes A/I = \bar{f}$ and $g(P) = IJ$ with $I + J = A$ and J is a local complete intersection ideal of height n. Then $(A/I) = C_n(P^*) = (A/IJ) = (A/I) + (A/J)$ and hence $(A/J) = 0$. By [5], J is a complete intersection ideal.

So, we have the following two exact sequences:

$$0 \to K \to A^n \xrightarrow{h} J \to 0 \quad \text{and} \quad 0 \to L \to P \xrightarrow{g} IJ \to 0.$$

Let $\wp_1, \wp_2, \ldots, \wp_r$ be the minimal prime ideal of A and for, $1 \leq i \leq r$, let m_i be a maximal ideal that contains $\wp_i$ and $m_i + IJ = A$. Write $J' = \bigcap_{i=1}^{r} m_i$. Now the inclusion $IJ \hookrightarrow J$ induces an isomorphism $IJ \otimes A/JJ' \approx J \otimes A/JJ'$ of projective A/JJ'-modules. Hence there is a isomorphism $\bar{\eta} : P/JJ'P \to A^n JJ'A^n$ such that the diagram

$$\begin{array}{ccc} (A/JJ')^n & \longrightarrow & J \otimes A/JJ' \\ {\scriptstyle \bar{\eta}}\uparrow{\scriptstyle \wr} & & \uparrow{\scriptstyle \wr} \\ P/JJ'P & \longrightarrow & IJ \otimes A/JJ' \end{array}$$

commutes. Let $\eta : P \to A^n$ be a lift of $\bar{\eta}$. Then $A^n = \eta(P) + JJ'A^n$. So, there is an a in A such that $(a, JJ') = A$ and $a(\text{cokernel}\,\eta) = 0$. Note that a is a non-zero divisor. Since we have surjective map $A^n/\eta(P) \to J/IJ \approx A/I$, note that a is in I. We have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & A^n & \xrightarrow{h} & J & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle \eta} & & \uparrow & & \\ 0 & \longrightarrow & M & \longrightarrow & P & \xrightarrow{g} & IJ & \longrightarrow & 0. \end{array}$$

By tensoring the top row by $A/(a)$, we get the exact sequence

$$0 \to K/aK \to A^n/aA^n \to J/aJ \approx A/(a) \to 0.$$

Thus K/aK is stably free $A/(a)$-module of rank $n-1$ and hence free by [9]. Let $x_1, \ldots, x_{n-1}$ in K be such that their images in K/aK form a base of K/aK.

Choose x_n in A^n such that $h(x_n) = b \equiv 1$ modulo (a). Modifying b by an element of Ja, we may assume that b is a non-zero divisor.

Let "bar" denote going "modulo (a)". Thus $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ is a base for $\bar{A}^n$. Let μ be the matrix $[x_1, \ldots, x_n]$ in $\mathbb{M}_n(A)$. Without loss, we assume that $\det \bar{\mu} = 1$.

We have $A^n = \sum_{i=1}^n Ax_i + a^2A^n = \sum_{i=1}^n Ax_i + a\eta(P)$. We also have the exact sequences

$$0 \to Q \to A^n \oplus P \xrightarrow{\phi=(\mu,a\eta)} A^n \to 0$$

and

$$0 \to P \xrightarrow{\binom{0}{1_P}} A^n \oplus P \xrightarrow{(1_{A^n},0)} A^n \to 0.$$

The map $\phi = (\mu, a\eta)$ sends $(\lambda_1, \ldots, \lambda_n, p)$ to $\sum_{i=1}^n \lambda_i x_i + a\eta(p)$. Since rank of P is bigger than or equal to n, by [9], there exist a τ in $SL(A^n \oplus P)$ such that $(\mu, a\eta) \circ \tau = (I_{A^n}, O)$ (See also [8]). Let $\tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ where $\alpha \in \mathrm{End}(A^n)$, $\beta \in \mathrm{Hom}(P, A^n)$, $\gamma \in \mathrm{Hom}(A^n, P)$, $\delta \in \mathrm{End}(P)$. It follows, $\mu\alpha + a\eta\gamma = I_{A^n}$ and $\mu\beta + a\eta\delta = 0$.

Again, let "bar" denote going "modulo a". Hence, we have $\bar{\mu}\bar{\alpha} = I_{\bar{A}^n}$ and $\bar{\mu}\bar{\beta} = 0$. Hence, $\det \bar{\alpha} = 1$ and $\bar{\beta} = 0$. Therefore, $\det \bar{\delta} = \det \bar{\tau} = 1$.

Let $\theta : Q \to P$ denote the restriction of the projection map $A^n \oplus P \to P$ and let $f' : P \to IJ$ be the map $f' = g\theta\tau_1$, where $\tau_1 : P \xrightarrow{\sim} Q$ is the isomorphism induced by τ. In fact $\tau_1(p) = \tau \begin{pmatrix} 0 \\ p \end{pmatrix} = \begin{pmatrix} \beta(p) \\ \delta(p) \end{pmatrix}$ for p in P. Thus $f' = g\delta$.

we now show that image of $f' = bI$ or equivalently, we prove that $\mathrm{Im}(g\theta) = bI$. For $(\lambda_1, \ldots, \lambda_n, p)$ in Q, we have $\sum_{i=1}^n \lambda_i x_i + a\eta(p) = 0$. Hence

$$0 = h(\lambda_n x_n + a\eta(p)) = \lambda_n b + ag(p).$$

Since $Aa + Ab = A$, $g(p) \in I \cap (b) = bI$. Thus $\mathrm{Im}(f') \subseteq bI$. Now consider λb, with λ in I. There is a p in P, with $h\eta(p) = g(p) = \lambda b$. So, $h(-\lambda x_n + \eta(p)) = 0$. Hence $-\lambda x_n + \eta(p) \in K \subseteq \sum_{i=1}^{n-1} Ax_i + aA^n \subseteq \sum_{i=1}^{n-1} Ax_i + \eta(P)$. Thus, we have $-\lambda x_n + \eta(p) = \sum_{i=1}^{n-1} \lambda_i x_i + \eta(p')$ for some p' in P and $\lambda_1, \ldots, \lambda_{n-1}$ in A. Note that $g(p') = h\eta(p') = 0$. Clearly, $z = (-\lambda_1 a, \ldots, -\lambda_{n-1} a, -\lambda a, p - p') \in Q$ and hence $\lambda b = g(p) = g\theta(z) \in \mathrm{Im}(f')$. This shows that $\mathrm{Im}(f') = Ib$.

Let f'' denote the composite map

$$f'' : P \xrightarrow{f'} Ib \approx I.$$

Let "$\sim$" denote going "modulo I". For some fixed base of $\widetilde{P}$, we have $\widetilde{\delta}$ is in $SL_n(\widetilde{A}) = E_n(\widetilde{A})$. By [1], there is an automorphism $\phi \in \mathrm{Aut}(P)$ such that $\widetilde{\phi} = \widetilde{\delta}$. Set $f = f''\phi^{-1}$. Then

$$\widetilde{f} = \widetilde{f'}\widetilde{\phi}^{-1} = \widetilde{g}\widetilde{\delta}\widetilde{\phi}^{-1} = \widetilde{g} = \bar{f}.$$

This completes the proof of Theorem 2.1. □

Taking I to be a complete intersection ideal of height n and $P = A^n$ in Theorem 2.1, we get the following.

Corollary 2.2. *Let A be a reduced affine algebra of dimension n over an algebraically closed field k. Assume $F^n K_0 A$ has no $(n-1)!$ torsion. Let I be a complete intersection ideal of height n. Then any base $\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_n$ of I/I^2 can be lifted to a set of generators of I.*

The following example shows that Corollary 2.2 fails if k is not algebraically closed.

Example 2.3. Let $A = \mathbb{R}[x, y, z]$ be the coordinate ring of the real two sphere $x^2 + y^2 + z^2 = 1$. Let $I = (x, y)$ in A. The generators $(-\bar{y}, \bar{z}\bar{x})$ of I/I^2 does not lift to generators of I.

Proof. Let $P = A^3/(x, y, z)$ be the projective A-module corresponding to the tangent bundle of the real two sphere. Suppose, if possible, $I = (f, g)$ where $\bar{f} = -\bar{y}$ and $\bar{g} = \bar{z}\bar{x}$ in I/I^2. (Here "bar" denotes going "modulo I").

Let η, η', η'' in $\operatorname{Ext}^1(I, A) \approx A/I$, respectively be the following extensions:

$$\eta : 0 \to A \xrightarrow{\binom{x}{y}} A^2 \xrightarrow{(-y,x)} I \to 0$$
$$\eta' : 0 \to A \xrightarrow{\binom{-g}{f}} A^2 \xrightarrow{(f,g)} I \to 0$$
$$\eta'' : 0 \to A \xrightarrow{\psi} P \xrightarrow{\phi} I \to 0$$

where ϕ is the map induced by the map $A^3 \to I$ that sends the standard base e_1, e_2, e_3, respectively, to $-y, x, 0$ and ψ is defined by $\psi(1) =$ image$(-ze_3)$ in P.

For a suitable $\alpha \in \operatorname{End}(A^2)$, we have the commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & A^2 & \xrightarrow{(-y,x)} & I & \longrightarrow & 0 \\ & & \downarrow d & & \downarrow \alpha & & \| & & \\ 0 & \longrightarrow & A & \longrightarrow & A^2 & \xrightarrow{(f,g)} & I & \longrightarrow & 0 \end{array}$$

where $d = \det(\alpha)$. By tensoring with A/I, we get the commutative diagram

$$\begin{array}{ccc} \bar{A}^2 & \xrightarrow{\sim} & I/I^2 = (-\bar{y}, \bar{x})\bar{A} \\ \downarrow \bar{\alpha} & & \| \\ \bar{A}^2 & \xrightarrow{\sim} & I/I^2 = (-\bar{y}, \bar{z}\bar{x})\bar{A}. \end{array}$$

Hence

$$\bar{\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & \bar{z} \end{pmatrix}$$

and $\det \bar{\alpha} = \bar{z}$. Thus $\eta' = z\eta$ in $\operatorname{Ext}^1(I, A)$.

On the other hand, we also have the commutative diagram

$$\begin{array}{ccccccccc} \eta : 0 & \longrightarrow & A & \longrightarrow & A^2 & \longrightarrow & I & \longrightarrow & 0 \\ & & \downarrow z & & \downarrow \beta & & \| & & \\ \eta'' : 0 & \longrightarrow & A & \longrightarrow & P & \longrightarrow & I & \longrightarrow & 0 \end{array}$$

where β is induced by the inclusion $A^2 \hookrightarrow A^3$ that sends (a, b) to $(a, b, 0)$. Hence $\eta'' = z\eta = \eta'$ in $\operatorname{Ext}^1(I, A)$. So, $P \approx A^2$, which contradicts the fact that P is not free. This completes the proof of Example 2.3. $\square$

Following Example 2.4 shows that Corollary 2.2 fails for ideals which are not local complete intersection of height n.

Example 2.4. Let $A = \mathbb{C}[X,Y]$ and let f be in A be such that $SK_1(A/(f)) \neq 0$; for example, take $f = X^3 + Y^3 - 1$ [7]. Let $I = Af$. Then $I/I^2 = Af/Af^2 \approx A/(f)$. Suppose every two generators of I/I^2 can be lifted to two generators of I.

It follows from the commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & A/I \\ f \big\downarrow \wr & & f \big\downarrow \wr \\ I & \longrightarrow & I/I^2 \end{array}$$

that any unimodular vector of length two over $A/(f)$ can be lifted to a unimodular vector of A of length two. This would imply that $SK_1(A/(f)) = 0$, which is a contradiction. This completes the proof of Example 2.4. $\square$

Let A be as in Theorem 2.1 and further assume that A is a regular domain of dimension n. Let P be a projective A-module of rank n and I be any ideal. Let $\bar{f} : P \twoheadrightarrow I/I^2$ be a surjective map. Theorem 2.6 gives a necessary and sufficient condition for the existence of a surjective map $f : P \twoheadrightarrow I$. Although, as the Example 2.4 suggests, we may not be able to lift $\bar{f}$ to a surjective map $f : P \twoheadrightarrow I$.

Before stating Theorem 2.6, we fix some notation.

Let A be a regular affine algebra of dimension n over an algebraically closed field k. Let P be a projective A-module of rank n and I be an ideal of A. Let $X = \operatorname{Spec} A$ and $\pi : \widetilde{X} \to X$ be the blow up of the ideal I and $I\mathcal{O} = \mathcal{O}_{\widetilde{X}}(1)$. Set

$$s(P,I) = \sum (-1)^i C_{n-i}(P)\pi_*(C_1(\mathcal{O}_{\widetilde{X}}(1))^i \cap \widetilde{X}).$$

Then we have the following lemma:

Lemma 2.5. *With notations as above, let $f \in P^* = \operatorname{Hom}(P,A)$ with $f(P) = I \cdot J$, where $J + I = A$ and J is a local complete intersection ideal of height n or $J = A$, then $\zeta(P,I) = \operatorname{cycle}(A/J)$. In particular, if f is surjective onto I, then $\zeta(P,I) = 0$.*

Proof. We have $\pi^* f : \pi^* P \to J\mathcal{O}_{\widetilde{X}}(1)$ is surjective. Hence, we get a surjective map $\phi^*(P)(-1) \to J\mathcal{O}_{\widetilde{X}}$. By [4],

$$\begin{aligned} C_n(\pi^*(P)(-1)) \cap \widetilde{X} &= \sum_{i=0}^{n} (-1)^i C_{n-i}(\pi^* P) \cap (C_1(\mathcal{O}_{\widetilde{X}}(1)))^i \cap \widetilde{X} \\ &= \text{cycle}\,(\mathcal{O}_{\widetilde{X}}/J\mathcal{O}_{\widetilde{X}}). \end{aligned}$$

Since $V(I) \cap V(J) = \emptyset$, applying π^*, to this equation above and using projection formula, we have $\zeta(P, I) = \text{cycle}\,(A/J)$. This completes the proof of Lemma 2.5. □

Now we state Theorem 2.6.

Theorem 2.6. *Let A be a regular affine algebra of dimension n over an algebraically closed field k. Let P be a projective A-module of rank n and I an ideal in A. Then the following conditions are equivalent:*

(1) *There is a surjective map $f : P \twoheadrightarrow I$ onto I.*

(2) *There is a surjective map $\bar{f} : P \twoheadrightarrow I/I^2$ and $\zeta(P, I) = 0$.*

Proof. Condition (1) implies condition (2) follows from Lemma 2.5. So, we prove (2) implies (1).

Take a general lift $f : P \to I$ of $\bar{f} : P \twoheadrightarrow I/I^2$. Then $f(P) = IJ$, where J is local complete intersection ideal of height n with $I + J = A$. Hence, by Lemma 2.5, $0 = \zeta(P, I) = \text{cycle}\,(A/J)$. By ((3.4), [5]), J is a complete intersection ideal. Hence by ((1.6), [5]), there is a surjective map $f' : P \twoheadrightarrow I$. This completes the proof of Theorem 2.6. □

3. Affine Algebras over $\bar{F}_p$

In this section we prove a stronger version (Theorem 3.2) of Theorems 2.1 and 2.6 when the ground filed is $\bar{F}_p$ (p is a prime number).

We begin with the following lemma.

Lemma 3.1. (See [10]) *Let A be an affine algebra of dimension $n \geq 1$, over $\bar{F}_p$. Let J be a complete intersection ideal of height n. Suppose $J = (a_1, \ldots, a_{n+1})A$. Then there exists $\lambda = (\lambda_1, \ldots, \lambda_{n+1})$ in A^{n+1} such that λ is unimodular and $\sum_{i=1}^{n+1} \lambda_i a_i = 0$.*

Proof. We prove by induction on n. We also note that we may modify the generators $(a_1, \ldots, a_{n+1})$ of J by any element in $GL_{n+1}(A)$.

for $n = 1$, we have $J = (h) = (a_1, a_2)$, where h is a non-zero divisor. Hence $a_1 = -\lambda_1 h$ and $a_2 = \lambda_2 h$ for some λ_1, λ_2 in A. It follows that (λ_1, λ_2) is unimodular and $\lambda_2 a_1 + \lambda_1 a_2 = 0$. Hence the proof in the case $n = 1$ is complete by taking $\lambda = (\lambda_2, \lambda_1)$.

Assume now $n \geq 2$ and $J = (h_1, \ldots, h_n)$. We assume that h_1 and h_n are non-zero divisors in A. Let "bar" denote going "modulo (h_n)". We have $\bar{J} = (\bar{h}_1, \ldots, \bar{h}_{n-1}) = (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_{n+1})$. By [2, 3], there are $\bar{\mu}_2, \ldots, \bar{\mu}_{n+1}$ in $\bar{A}$ such that $\bar{J} = (\bar{a}_2 + \bar{\mu}_2 \bar{a}_1, \ldots, \bar{a}_{n+1} \bar{\mu}_{n+1} \bar{a}_1)$. Hence there exists $\bar{\varepsilon}$ in $E_{n+1}(\bar{A})$ such that $(\bar{a}_1, \ldots, \bar{a}_{n+1})\bar{\varepsilon} = (\bar{h}_1, \bar{a}_2 + \bar{\mu}_2 \bar{a}_1, \ldots, \bar{a}_{n+1} + \bar{\mu}_{n+1} \bar{a}_1)$. Lifting $\bar{\varepsilon}$ to ε in $E_{n+1}(A)$ and replacing $(a_1, \ldots, a_{n+1})$ by $(a_1, \ldots, a_{n+1})\varepsilon$, we may assume that $a_1 = h_1 + \mu h_n$ for some $\mu \in A$. Modifying $h_1 + \mu h_n$ by and element of $h_n(a_2, \ldots, a_{n+1})$, we may assume that $a_1 = h_1 + \mu h_n$ is not a zero divisor. Replacing h_1 be $h_1 + \mu h_n$, we may assume $a_1 = h_1$.

Now let "$\sim$" denote going "modulo h_1". So, $\widetilde{J} = (\widetilde{h}_2, \ldots, \widetilde{h}_n) = (\widetilde{a}_2, \ldots, \widetilde{a}_{n+1})$. Hence by induction hypothesis there exists $(\widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_{n+1})$ unimodular in $\widetilde{A}$ such that $\sum_{i=2}^{n+1} \widetilde{\lambda}_2 \widetilde{a}_i = 0$.

Since by [12], stable range of $A \leq n$, $(\widetilde{\lambda}_2, \ldots, \widetilde{\lambda}_{n+1})$ can be lifted to a unimodular vector $(\lambda_2, \ldots, \lambda_{n+1})$ in A^n. Hence we have $\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_{n+1} a_{n+1} = 0$ for some λ_1 in A. This completes the proof of Lemma 3.1. □

Theorem 3.2. *Let A be a reduced affine $\bar{F}_p$-algebra of dimension n. Suppose*

(i) $n \geq 3$ *or*

(ii) $n = 2$ *and A is a regular affine domain.*

Let P be a projective A-module of rank n and I any ideal of A. Then any surjective map $\bar{f} : P \twoheadrightarrow I/I^2$ lifts to a surjective map $f : P \twoheadrightarrow I$.

Proof. Let $f' : P \to I$ be a general lift of $\bar{f}$. Then as before, $f'(P) = IJ$ with $I + J = A$ and J is a product of distinct smooth maximal ideals of height n. The hypothesis (i) and (ii) imply that (see [6]) $F^n K_0 A = 0$ and also that J is a complete intersection ideal. Since $F^n K_0 A = 0$, we have $P \approx P' \oplus Ap_n$ with $\operatorname{rank} P' = n-1$. As f' induces an isomorphism of $P/JP \xrightarrow{\sim} J/J^2$, there exist $p_1, \ldots, p_{n-1}$ in P' such that $f'(p_1), \ldots, f'(p_n)$ is a base for J modulo J^2.

Let

$$\mathcal{N} = \mathcal{O}(p_1 \Lambda p_2 \Lambda \cdots \Lambda p_n) = \operatorname{Image}\left\{\Lambda^n P^* \xrightarrow{p_1 \Lambda p_2 \Lambda \cdots \Lambda p_n} A\right\}.$$

Let $V(\mathcal{I})$ be the singular locus of $\operatorname{Spec} A$. We have $J + I\mathcal{I}\mathcal{N} = A$.

Case 1. $n \geq 3$.

By Swan's Bertini's theorem (see for example, (2.4), [5]) there exists an h in J such that $h - 1$ is in $I\mathcal{I}\mathcal{N}$ and A/Ah is smooth of dimension $n - 1 \geq 2$.

Let "bar" denote going "modulo (h)". Then, since $F^{n-1} K_0 A = 0$ and $\bar{J}$ is a complete intersection of height $n-1$. Also $\bar{P}$ is $\bar{A}$-free, with base $\bar{p}_1, \ldots, \bar{p}_n$ and $\bar{f}'(\bar{P}) = \bar{J}$.

So, by Lemma 3.1 there exists an unimodular $(\bar{\lambda}_1, \ldots, \bar{\lambda}_n) \in \bar{A}^n$ such that $\sum_{i=1}^n \bar{\lambda}_i f'(\bar{p}_i) = 0$. Since $\sum_{i=1}^n \bar{\lambda}_i \bar{p}_i$ is unimodular in $\bar{P}$, by [6], there exists a p' in P unimodular with $\bar{p}' = \sum_{i=1}^n \bar{\lambda}_i \bar{p}_i$. Thus we have $P = P'' \oplus Ap'$ and $f'(\bar{p}') = 0$ i.e. $f'(p') = ah$, for some a in A. Since J is a smooth ideal of height $n = \operatorname{rank} P$, it is easy to see that $Aa + J = A$. Since $ah = f'(p') \in IJ$ and $h - 1 \in I$, it follows that $a \in I$. Since $I/f'(P) = I/IJ \approx A/J$, it follows that $I = f'(P) + Aa$.

Define $f : P \to I$ by setting $f|_{P''} = f'|_{P''}$ and $f(p') = a$. Then f is surjective and since $h - 1 \in I$, it is clear that f is a lift of $\bar{f}$. This completes the proof of Case 1.

Case 2. $n = 2$.

Let $J = (h_1, h_2)$ and let "bar" denote going "modulo $\mathcal{N}\mathcal{I}$". Then $(\bar{h}_1, \bar{h}_2) \in \bar{A}^2$ is unimodular. By [12] or [6], there is $(h_1', h_2') \in A^2$ unimodular such that $\bar{h}_1' = \bar{h}_i$ for $i = 1, 2$. Let $\alpha \in SL_2 A$ be such that $(h_1', h_2')\alpha = (0, 1)$. Replacing (h_1, h_2) by $(h_1, h_2)\alpha$, we assume $\bar{h}_2 = 1$.

Let "$\sim$" denote going "modulo (h_2)". Then $\widetilde{P}$ is $\widetilde{A}$-free with base $\widetilde{p}_1, \widetilde{p}_2$ and $\widetilde{J} = (\widetilde{h}_1) = (f'(\widetilde{p}_1), f'(\widetilde{p}_2))$. By Lemma 3.1, choose $(\widetilde{\lambda}_1, \widetilde{\lambda}_2) \in \widetilde{A}^2$ unimodular with $\widetilde{\lambda}_1 f'(\widetilde{p}_1) + \widetilde{\lambda}_2 f'(\widetilde{p}_2) = 0$. Now, we can complete the proof as in Case 1, by lifting $\widetilde{\lambda}_1 \widetilde{p}_1 + \widetilde{\lambda}_2 p_2$ to an unimodular element p' in P. This completes the proof of Theorem 3.2. □

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