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Corresponding Author	Family Name	**Mandal**
	Particle	
	Given Name	**Satya**
	Suffix	
	Division	
	Organization	University of Kansas
	Address	Lawrence, KS, 66045, USA
	Phone	
	Fax	
	Email	mandal@ku.edu
	URL	
	ORCID	
Author	Family Name	**Mishra**
	Particle	
	Given Name	**Bibekananda**
	Suffix	
	Division	
	Organization	University of Kansas
	Address	Lawrence, KS, 66045, USA
	Phone	
	Fax	
	Email	bibekanandamishra@ku.edu
	URL	
	ORCID	

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Abstract

For a projective A-module P, with $n = rank(P) \geq 2$, the Homotopy obstruction sets $\pi_0\left(\mathcal{LO}(P)\right)$ were defined, in [6], to detect whether P has a free direct summand or not. These sets have a structure of an abelian monoid, under suitable regularity and other conditions. In this article, we provide some further perspective on these sets $\pi_0\left(\mathcal{LO}(P)\right)$. In particular, under similar regularity and other conditions, we prove that if P, Q are two projective A-modules, with $rank(P) = rank(Q) = d$ and $\det(P) \cong \det Q$, then $\pi_0\left(\mathcal{LO}(Q)\right) \cong \pi_0\left(\mathcal{LO}(P)\right)$.

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ORIGINAL RESEARCH

Some perspectives on homotopy obstructions

Satya Mandal · Bibekananda Mishra

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Abstract For a projective A-module P, with $n = rank(P) \geq 2$, the Homotopy obstruction sets $\pi_0(\mathcal{LO}(P))$ were defined, in [6], to detect whether P has a free direct summand or not. These sets have a structure of an abelian monoid, under suitable regularity and other conditions. In this article, we provide some further perspective on these sets $\pi_0(\mathcal{LO}(P))$. In particular, under similar regularity and other conditions, we prove that if P, Q are two projective A-modules, with $rank(P) = rank(Q) = d$ and $\det(P) \cong \det Q$, then $\pi_0(\mathcal{LO}(Q)) \cong \pi_0(\mathcal{LO}(P))$.

Keywords Projective modules · Chow groups

1 Introduction

Throughout this article A will denote a commutative noetherian ring, with $\dim A = d \geq 2$, and $A\ [T]$ will denote the polynomial ring in one variable T. For an A-module M, denote $M[T] := M \otimes A[T]$. Likewise, for a homomorphism $f : M \longrightarrow N$ of A-modules, $f[T] := f \otimes A[T]$. Also, P will denote a projective A-module with $rank(P) = n$. By a local P-orientation we mean a pair (I, ω) where I is an ideal and $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is a surjective map. To facilitate discussions we introduce the following notations:

$$\begin{cases} \mathcal{LO}(P) = \{(I, \omega) : (I, \omega) \text{ is a local } P\text{-orientation}\} \\ \widetilde{\mathcal{LO}}(P) = \{(I, \omega) \in \mathcal{LO}(P) : height(I) = n \text{ or } I = A\} \end{cases} \tag{1}$$

Substituting $T = 0, 1$ we obtain two maps

$$\mathcal{LO}(P) \xleftarrow{T=0} \mathcal{LO}(P[T]) \xrightarrow{T=1} \mathcal{LO}(P)$$

This generates a (chain homotopy) equivalence relation $\mathcal{LO}(P)$. The (Nori) Homotopy obstruction set $\pi_0(\mathcal{LO}(P))$ was defined [6] to be the set of all equivalence classes. Recall, an obstruction class $\varepsilon(P) \in$

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S. Mandal (✉) · B. Mishra
University of Kansas, Lawrence, KS 66045, USA
E-mail: mandal@ku.edu

B. Mishra
E-mail: bibekanandamishra@ku.edu

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$\pi_0(\mathcal{LO}(P))$ was also defined. All these emanates from the Homotopy Question [6, (1.1)] posed by M. V. Nori, to detect whether P would have a free direct summand or not. For further introductory comments and background on Homotopy obstructions, the readers are directed to [6, 7].

For the rest of this introductory section, we assume that A is a regular ring, containing a field k, with $1/2 \in k$. It was proved in [6] that, $\pi_0(\mathcal{LO}(P))$ has structure of an abelian monoid, when $2n \geq d+2$. Further, if $P \cong P_0 \oplus A$, then $\pi_0(\mathcal{LO}(P))$ would be a group.

It was also proved [6] that, if A is essentially smooth over an infinite perfect field k, and $2n \geq d+3$, then

$$\varepsilon(P) = \text{neutral} \Longleftrightarrow P \cong Q \oplus A$$

In this article, we provide some further perspectives on the additive structure of these obstruction sets $\pi_0(\mathcal{LO}(P))$, by proving that some of them are isomorphic and by defining natural set theoretic maps to Chow groups.

Assume $n = rank(P) \geq 2$. Restricting the above equivalence relation to $\widetilde{\mathcal{LO}}(P)$, we obtain a set of equivalence classes, to be denoted by $\pi_0\left(\widetilde{\mathcal{LO}}(P)\right)$. We prove that the natural map $\pi_0\left(\widetilde{\mathcal{LO}}(P)\right) \xrightarrow{\sim} \pi_0(\mathcal{LO}(P))$ is, in deed, an isomorphism. We establish this in section 2.

This alternate description of the same set $\pi_0(\mathcal{LO}(P))$ would be more close to the original statement of the Homotopy question [6, (1.1)], and would be of key importance for technical purposes.

The main results in this article are in section 3, that we discuss in this paragraph. Let P, Q be two projective A-modules with $rank(P) = rank(Q) = d$ and let $\imath : \Lambda^d Q \xrightarrow{\sim} \Lambda^d P$ be an isomorphism of the determinants. Under the hypotheses above, we prove (Theorem 3.6) that there is a natural monoid isomorphism $\chi(\imath) : \pi_0(\mathcal{LO}(P)) \xrightarrow{\sim} \pi_0(\mathcal{LO}(Q))$. Therefore, when $rank(P) = d$, we have the following:

There is a natural monoid isomorphism

$$\pi_0(\mathcal{LO}(P)) \cong \pi_0\left(\mathcal{LO}(\Lambda^d P \oplus A^{d-1})\right)$$

Since $\pi_0\left(\mathcal{LO}(\Lambda^d P \oplus A^{d-1})\right)$ is an abelian group, so is $\pi_0(\mathcal{LO}(P))$.

Therefore, we observe that, for $\pi_0(\mathcal{LO}(P))$ to have a group structure, it is not necessary that $P \cong P_0 \oplus A$.

If A is essentially smooth over an infinite perfect field k, with $d \geq 3$, it follows that $\pi_0(\mathcal{LO}(P)) \cong E^d(A, \Lambda^d P)$ where $E^d(A, \Lambda^d P)$ denotes the Euler class group, as defined in [1] (also see [2, 6–8]).

Under the same regularity conditions, as above, in section 4, we establish a natural set theoretic map $\pi_0(\mathcal{LO}(P)) \longrightarrow CH^n(A)$, where $n = rank(P) \geq 2$ and $CH^n(A)$ denotes the Chow group of codimension n cycles [4]. With suitable modifications, the regularity hypothesis can be relaxed, for the existence of such a map (Proposition 4.2).

Remark In a subsequent article [5], we establish a (set theoretic) map

$$\Theta_P : \pi_0(\mathcal{LO}(P)) \longrightarrow \widetilde{CH}^n(A, \Lambda^n P)$$

where $\widetilde{CH}^n(A, \Lambda^n P)$ denotes the Chow Witt groups [3].

2 Alternate Description of the Obstructions

Let A be a noetherian commutative ring, with $\dim A = d$ and P be a projective A-module with $rank(P) = n$. In [6], the Homotopy Obstruction sets $\pi_0(\mathcal{LO}(P))$ were defined, along with several other description of the same. In this section, we recall the essential elements of the definition of $\pi_0(\mathcal{LO}(P))$, and give another description of the same, which is of some key technical importance.

Definition 2.1 Let A be a noetherian commutative ring, with $\dim A = d$ and P be a projective A-module with $rank(P) = n$. By a **local P -orientation**, we mean a pair (I, ω) where I is an ideal of A and $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is a surjective homomorphism. We will use the same notation ω for the map $\frac{P}{IP} \twoheadrightarrow \frac{I}{I^2}$, induced by ω. Denote

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$$\begin{cases} \mathcal{LO}(P) = \{(I,\omega) : (I,\omega) \text{ is a local } P \text{ orientation}\} \\ \widetilde{\mathcal{LO}}(P) = \{(I,\omega) \in \mathcal{LO}(P) : height(I) = n \text{ or } I = A\} \end{cases} \tag{2}$$

The obstruction set $\pi_0(\mathcal{LO}(P))$ was defined [6], by the push forward diagram:

$$\begin{array}{ccc} \mathcal{LO}(P[T]) & \xrightarrow{T=0} & \mathcal{LO}(P) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \mathcal{LO}(P) & \longrightarrow & \pi_0(\mathcal{LO}(P)) \end{array} \quad \text{in} \quad \underline{Sets}. \tag{3}$$

While one would like to define $\pi_0\left(\widetilde{\mathcal{LO}}(P)\right)$ similarly, note that substitutions $T = 0, 1$ would not yield any map from $\widetilde{\mathcal{LO}}(P[T])$ to $\widetilde{\mathcal{LO}}(P)$. However, note that the definition of $\pi_0(\mathcal{LO}(P))$ by push forward diagram (3), is only an alternate way of saying the following:

For $(I_0,\omega_0), (I_0,\omega_0) \in \mathcal{LO}(P)$, we write $(I_0,\omega_0) \sim (I_0,\omega_0)$, if

$$\exists \quad (I,\omega) \in \mathcal{LO}(P[T]) \quad \ni \quad \begin{cases} (I(0),\omega(0)) = (I_0,\omega_0) \\ (I(1),\omega(1)) = (I_1,\omega_1) \end{cases}$$

The homotopy relation $\sim$ generates an equivalence relation on $\mathcal{LO}(P)$, which we denote by $\approx$.
The above definition (3) means, $\pi_0(\mathcal{LO}(P))$ is the set of all equivalence classes in $\mathcal{LO}(P)$.

The restriction of the relation $\approx$ on $\widetilde{\mathcal{LO}}(P) \subseteq \mathcal{LO}(P)$, defines an equivalence relation on $\widetilde{\mathcal{LO}}(P)$. Define $\pi_0\left(\widetilde{\mathcal{LO}}(P)\right)$ to be the set of all equivalence classes in $\widetilde{\mathcal{LO}}(P)$. It follows, that there is natural map

$$\varphi : \pi_0\left(\widetilde{\mathcal{LO}}(P)\right) \longrightarrow \pi_0(\mathcal{LO}(P))$$

Proposition 2.2 *Let A and P be as in (2.1). Then,*

The map φ is surjective.
If A is a regular ring containing is field k, with $1/2 \in k$, then φ is a bijection.
Let A be as in (2). Then $\sim$ is an equivalence relation on $\widetilde{\mathcal{LO}}(P)$.

Proof Let $x = [(I,\omega)] \in \pi_0(\mathcal{LO}(P))$. By application of the Involution operation [6, Section 5] (*or moving lemma argument*) twice, we can assume $height(I) \geq n$. This establishes that φ is surjective. Now, assume A is as in (2). Suppose $x_0, x_1 \in \pi_0\left(\widetilde{\mathcal{LO}}(P)\right)$, and $\varphi(x_0) = \varphi(x_1)$. Then, for $i = 0, 1$, we have $x_i = [(I_i,\omega_i)]$, for some $(I_i,\omega_i) \in \widetilde{\mathcal{LO}}(P)$. By [6, Corollary 3.2], there is a homotopy $H(T) = (J,\Omega) \in \mathcal{LO}(P[T])$ such that $H(0) = (I_0,\omega_0)$ and $H(1) = (I_1,\omega_1)$. Therefore, $x_0 = x_1$. This establishes (2) and (3) follows by the same argument. The proof is complete. □

Remark 2.3 We record the following, with notations as in (2.1):

Suppose $(I_0,\omega_0), (I_1,\omega_1) \in \widetilde{\mathcal{LO}}(P)$ and $(I_0,\omega_0) \approx (I_1,\omega_1)$. By definition there is homotopy $H(T) = (I,\omega) \in \mathcal{LO}(P[T])$ such that $H(0) = (I_0,\omega_0)$ and $H(1) = (I_1,\omega_1)$. By moving Lemma argument, similar to [6, Lemma 4.5], we can assume that $H(T) \in \widetilde{\mathcal{LO}}(P[T])$.
Assume A is a Cohen Macaulay ring. Then, $\widetilde{\mathcal{LO}}(P)$ is in bijection with the set

$$\left\{(I,\omega) \in \mathcal{LO}(P) : height(I) = n,\ \omega : \frac{P}{IP} \xrightarrow{\sim} \frac{I}{I^2} \text{ is an isomorphism}\right\} \cup \{(A,0)\}$$

3 Isomorphisms of $\pi_0(\mathcal{LO}(P))$: the top rank case

Under usual regularity hypotheses, in this section we prove that, the obstruction sets $\pi_0(\mathcal{LO}(P))$ are naturally isomorphic, when $rank(P) = d$

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Definition 3.1 Let A be a regular ring, containing a field k, $1/2 \in k$, and $\dim A = d \geq 2$. Let P, Q be projective A-modules with $rank(P) = rank(Q) = d$. Assume they have isomorphic determinant and $\iota : \Lambda^d Q \xrightarrow{\sim} \Lambda^d P$ is an isomorphism. We define a natural map

$$\chi(\iota) : \pi_0(\mathcal{LO}(P)) \longrightarrow \pi_0(\mathcal{LO}(Q)) \qquad \text{as follows:}$$

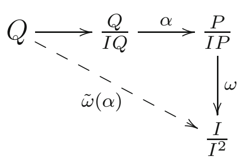
By Proposition 2.2, it would be enough to establish a natural map

$$\chi(\iota) : \pi_0\left(\mathcal{LO}^{\geq n}(P)\right) \longrightarrow \pi_0\left(\mathcal{LO}^{\geq n}(Q)\right)$$

Let $(I, \omega) \in \widetilde{\mathcal{LO}}(P)$ with $height(I) = d$. We use the same notation $\omega : \frac{P}{IP} \xrightarrow{\sim} \frac{I}{I^2}$ for the isomorphism, induced by $\omega : P \twoheadrightarrow \frac{I}{I^2}$. Let

$$\alpha : \frac{Q}{IQ} \xrightarrow{\sim} \frac{P}{IP} \quad \text{be an isomorphism,} \quad \ni \quad \det(\alpha) = \iota \otimes \frac{A}{I}. \tag{3}$$

Let $\tilde{\omega}(\alpha)$ be defined by the commutative diagram



and define

$$\begin{cases} \chi_0(\iota)(I, \omega) = (I, \tilde{\omega}(\alpha)) \in \mathcal{LO}(Q) \\ \chi_1(\iota)(I, \omega) = [(I, \tilde{\omega}(\alpha))] \in \pi_0\left(\mathcal{LO}(Q)\right). \end{cases}$$

For $x \in \pi_0(\mathcal{LO}(P))$, we can write $x = [(I, \omega)]$, for some $(I, \omega) \in \mathcal{LO}(P)$, with $height(I) \geq d$. We define,

$$\chi(\iota)(x) = \begin{cases} \chi_1(\iota)(I, \omega) \in \pi_0(\mathcal{LO}(Q)) & \textit{if height}(I) = d \\ [(A, 0)] \in \pi_0(\mathcal{LO}(Q)) & \textit{if } I = A \end{cases} \tag{4}$$

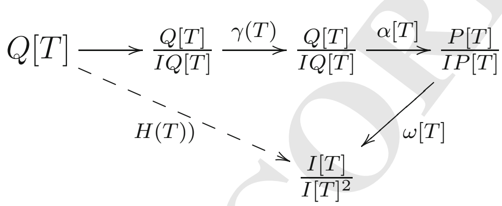
Subsequently, we prove that $\chi_1(\iota)(I, \omega)$ is independent of the choice of α, and that $\chi(\iota) : \pi_0(\mathcal{LO}(P)) \longrightarrow \pi_0(\mathcal{LO}(Q))$ is well defined.

Lemma 3.2 *With notations as in (3.1), let $(I, \omega) \in \mathcal{LO}(P)$, with $height(I) = d$. Then, $\chi_1(\iota)(I, \omega)$ is independent of the choice of α.*

Proof Let α be as in (3), and

$$\beta : \frac{Q}{IQ} \xrightarrow{\sim} \frac{P}{IP} \quad \text{be another isomorphism,} \quad \ni \quad \det(\beta) = \iota \otimes \frac{A}{I}.$$

Then, $\det(\alpha^{-1}\beta) = 1$. Since $\dim\left(\frac{A}{I}\right) = 0$, the map $\alpha^{-1}\beta$ is an elementary matrix, with respect to any choice of basis of $\frac{Q}{IQ}$. Therefore, there is an elementary automorphism $\gamma(T) \in EL\left(\frac{Q[T]}{IQ[T]}\right)$, such that $\gamma(0) = Id$ and $\gamma(1) = \alpha^{-1}\beta$. Consider $(I[T], H(T)) \in \mathcal{LO}(Q[T])$, where $H(T)$ is defined by the commutative diagram:



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Then with $T = 0, 1$, we have commutative diagrams

$$\begin{array}{ccccc} Q & \longrightarrow & \frac{Q}{IQ} & \xrightarrow{1} & \frac{Q}{IQ} \\ & \searrow^{H(0)} & & & \downarrow \omega\alpha \\ & & & & \frac{I}{I^2} \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} Q & \longrightarrow & \frac{Q}{IQ} & \xrightarrow{\alpha^{-1}\beta} & \frac{Q}{IQ} \\ & \searrow^{H(1)} & & & \downarrow \omega\alpha \\ & & & & \frac{I}{I^2} \end{array}$$

This completes the proof. ☐

Lemma 3.3 *With notations as in (3.1), the map* $\chi(\imath) : \pi_0(\mathcal{LO}(P)) \longrightarrow \pi_0(\mathcal{LO}(Q))$, *as defined by (4), is well defined.*

Proof Let $x = [(I, \omega)] = [(I_1, \omega_1)] \in \pi_0(\mathcal{LO}(P))$, with $height(I) \geq d$ and $height(I_1) \geq d$. Since the homotopy is an equivalence relation [6, Corollary 3.2], there is a homotopy $\mathcal{H}(T) = (J, \omega_J) \in \mathcal{LO}(P[T])$ such that

$$\mathcal{H}(0) = (J(0), \omega_J(0)) = (I, \omega), \qquad \text{and} \qquad \mathcal{H}(1) = (J(1), \omega_J(1)) = (I_1, \omega_1)$$

By Moving Lemma [6, Lemma 4.5], we can assume that $height(J) \geq d$. Since $\dim\left(\frac{A[T]}{J}\right) = 1$, it follows, there is an isomorphism

$$\alpha : \frac{Q[T]}{JQ[T]} \xrightarrow{\sim} \frac{P[T]}{JP[T]} \quad \ni \quad \det(\alpha) = \imath \otimes \frac{A[T]}{J}$$

Now define $\mathcal{H}_Q(T) = (J, \Omega_Q)$, where Ω_Q is defined as follows:

$$\begin{array}{ccccc} Q[T] & \longrightarrow & \frac{Q[T]}{JQ[T]} & \xrightarrow{\alpha} & \frac{P[T]}{JP[T]} \\ & \searrow^{\Omega_Q} & & & \downarrow \omega_J \\ & & & & \frac{J}{J^2} \end{array}$$

It follows
$$\begin{cases} \chi_1(\iota)(I, \omega) = [(I, \tilde{\omega}(\alpha(0)))] = [(J(0), \Omega_Q(0))] \\ \chi_1(\iota)(I_1, \omega_1) = [(I_1, \tilde{\omega}_1(\alpha(1)))] = [(J(1), \Omega_Q(1))] \end{cases}$$

The proof is complete. ☐

Proposition 3.4 *With notations as in (3.1), the map* $\chi(\imath)$ *is a monoid homomorphism.*

Proof Recall, in this case, $\pi_0(\mathcal{LO}(P))$ has an additive (monoid) structure [6, Theorem 6.5]. Let $x = [(I_1, \omega_1)], y = [(I_2, \omega_2) \in \pi_0(\mathcal{LO}(P))$. We can assume $I_1 + I_2 = A$ and $height(I_i) \geq d$ for $i = 1, 2$. If $I_1 = A$ or $I_2 = A$, then by definition (4), we have $\chi(\imath)(x + y) = \chi(\imath)(x) + \chi(\imath)(y)$. So, assume $height(I_1) = height(I_2) = d$. For $i = 1, 2$ let

$$\alpha_i : \frac{Q}{I_i Q} \xrightarrow{\sim} \frac{P}{I_i P} \quad \text{be isomorphisms} \ni \quad \det(\alpha_i) = \imath \otimes \frac{A}{I_i}.$$

Let $\alpha : \frac{Q}{I_1 I_2 Q} \xrightarrow{\sim} \frac{P}{I_1 I_2 P}$ be the isomorphism obtained by combining α_1 and α_2. Now, it follows that $\chi(\imath)(x) + \chi(\imath)(y)$ is obtained by combining $(I_1, \alpha_1\omega_1))$ and $(I_2, \alpha_2\omega_2))$, which is same as $\chi(\imath)(x + y)$. The proof is complete. ☐

Proposition 3.5 *Use the notations as in Definition 3.1. Then, Let Q' be another another projective A module with* $rank(Q) = d$ *and let* $\zeta : \Lambda^d Q' \xrightarrow{\sim} \Lambda^d Q$ *be an isomorphism. Then, we have* $\chi(\imath\zeta) = \chi(\zeta)\chi(\imath)$.

Proof Follows from Definition 3.1. ☐

Theorem 3.6 *Use the notations as in Definition 3.1. Then, the map* $\chi(\imath) : \pi_0(\mathcal{LO}(P)) \longrightarrow \pi_0(\mathcal{LO}(Q))$ *is a monoid isomorphism.*

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In particular, taking $Q = \Lambda^d Q \oplus A^{d-1}$, *it follows* $\pi_0(\mathcal{LO}(P))$ *is a group.*

Proof It follows from Proposition 3.5, that $\chi(\imath)\chi(\imath^{-1}) = Id$ and $\chi(\imath^{-1})\chi(\imath) = Id$, Therefore, $\chi(\imath)$ is an isomorphism. Recall $\pi_0(\mathcal{LO}(\Lambda^d Q \oplus A^{d-1}))$ is a group [6, Theorem 6.11], which settles the latter part. □

Remark 3.7 Use the notations as in Definition 3.1. Theorem 3.6 asserts that $\pi_0(\mathcal{LO}(P))$ has a group structure, even when P does not have a free direct summand. However, it remains open, whether $\pi_0(\mathcal{LO}(P))$ would fail to be a group, for some projective A-module P, with $2rank(P) \geq d + 2$.

Corollary 3.8 *Let* A *be an essentially smooth ring over an infinite perfect field* k, *with* $1/2 \in k$, *and* $\dim A = d \geq 3$. *Let* Q *be a projective* A*-module, with* $rank(Q) = n$. *Let* $P = \Lambda^d Q \oplus A^{d-1}$. *Recall the definition of the Euler class groups* $E(P) \cong E^d(A, \Lambda^d Q)$, *as in* [1, 6]. *The Euler class* $e(Q) \in E^d(A, \Lambda^d Q)$, *was also defined in* [1]. *We have a commutative diagram of isomorphisms*

$$\begin{array}{ccc} \pi_0\left(\mathcal{LO}(Q)\right) & \xrightarrow{\chi(\imath)} & \pi_0\left(\mathcal{LO}(\Lambda^d Q \oplus A^{d-1})\right) \\ & \searrow & \wr\downarrow\psi \\ & & E^d(A, \Lambda^d Q) \end{array}$$

where the horizontal map $\chi(\imath)$ is as in (3.1), and the vertical isomorphism was defined in [6, §7]. We also have

$$\psi\chi(\varepsilon(Q)) = e(Q)$$

where $\varepsilon(Q) := [(0,0)] \in \pi_0(\mathcal{LO}(Q))$ denotes the homotopy obstruction class of Q, as defined in [6].

Proof Let $f : Q \twoheadrightarrow I$ be a surjective homomorphism, where I is an ideal of height d. Let $\omega : Q \twoheadrightarrow I/I^2$ is induced by f. Then, by definition $\varepsilon(Q) = [(I, \omega)]$. Rest of the proof follows from definition of Euler class $e(Q)$. This completes the proof. □

4 Natural maps to Chow groups

The Chow groups of codimension n cycles will be denoted by $CH^n(A)$ (see [4, §1.3]). For an ideal $J \subseteq A$, $cycle(J)$ would denote the cycle of the closed set $V(J)$ (see [4, §1.5]).

Definition 4.1 Assume A is a Cohen Macaulay ring, with $\dim A = d$. Also assume $\dim A_{\mathfrak{m}} = d$ for all $\mathfrak{m} \in \max(\mathfrak{A})$. As before, P is a projective A-module with $rank(P) = n$. Define a map

$$\ell_P^0 : \widetilde{\mathcal{LO}}(P) \longrightarrow CH^n(A) \quad \text{by} \quad \ell_P^0(I, \omega) = cycle(I) \in CH^n(A)$$

Now define the map

$$\ell_P : \pi_0\left(\widetilde{\mathcal{LO}}(P)\right) \longrightarrow CH^n(A) \quad \text{by} \quad \ell_P[(I, \omega)] = cycle(I) \in CH^n(A).$$

We prove next that ℓ_P is a well defined set theoretic map.

Proposition 4.2 *Use the notations, as in (4.1). Then, the map*

$$\ell_P : \pi_0\left(\widetilde{\mathcal{LO}}(P)\right) \longrightarrow CH^n(A)$$

is a well defined set theoretic map.

Proof For $t \in A$, let $\tau_t : CH^n(A[T]) \longrightarrow CH^n(A)$ denote the map induced by substitution $T = t$. Let $\pi^* : CH^n(A) \longrightarrow CH^n(A[T])$ denote the pull back map. Then, for any $t \in A$, we have $\tau_t = (\pi^*)^{-1}$ [4, Cor 6.5, pp. 111].

Let $(I_0, \omega_0), (I_1, \omega_1) \in \widetilde{\mathcal{LO}}(P)$ and assume $(I_0, \omega_0) \sim (I_1, \omega_1)$. We prove $cycle(I_0) = cycle(I_1) \in CH^n(A)$. There is a homotopy $H(T) = (J, \omega) \in \mathcal{LO}(P[T])$ such that $H(0) = (I_0, \omega_0)$ and $H(1) = (I_1, \omega_1)$. By Moving Lemma Argument (see Remark 2.3), we can assume $height(J) \geq n$. Now, we have

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$$\begin{cases} \ell_P^0(I_0, \omega_0) = cycle(J(0)) = \tau_0(cycle(J)) = (\pi^*)^{-1}(cycle(J)) \\ \ell_P^0(I_1, \omega_1) = cycle(J(1)) = \tau_1(cycle(J)) = (\pi^*)^{-1}(cycle(J)) \end{cases} \tag{5}$$

This completes the proof. □

Theorem 4.3 *Suppose A is a regular ring, containing a filed k, with $1/2 \in k$, and $\dim A = d$. Let P be a projective A-module with $rank(P) = n$. Then,*

By Proposition 2.2, $\pi_0(\mathcal{LO}^{\geq n}(P)) \xrightarrow{\sim} \pi_0(\mathcal{LO}(P))$ is a bijection. Therefore, ℓ_P defined in (4.1), defines a set theoretic map

$$\ell_P : \pi_0(\mathcal{LO}(P)) \longrightarrow CH^n(A)$$

Further,

$$\ell_P(\varepsilon(P)) = C^n(P^*) \in CH^n(A)$$

where $\varepsilon(P) = [(0,0)] \in \pi_0(\mathcal{LO}(P))$ denotes the homotopy obstruction class of P, and $C^n(P^*)$ denotes the top Chern class of the dual P^*.

Assume $2n \geq d + 2$. Then, ℓ_P is an additive map.

Proof For the last statement, recall under the hypotheses $\pi_0(\mathcal{LO}(P))$, has structure of an abelian monoid. Rest of the proofs are obvious. □

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