Perfect Modules and K-Theory localization sequences

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Prelude

- Algebraic K-Theory may mean different things to **different cohort**. Few experts know more than a few facets of Algebraic K-Theopry.
- Classically, for a commutative ring A, three groups $K_0(A)$, $K_1(A)$, $K_2(A)$ were defined. Some exact sequences of such groups were also defined.

Prelude

- Subsequently, the leaders tried to define higher K-groups $K_n(A)$ $\forall n \geq 0$, and establish all the usual and expected exact sequences.
- Eventually, Daniel Quillen defined these groups $K_n(A)$ $\forall n \geq 0$, and more. I talk about Quillen K-Theory.

Basic Notations

- Throughout, X will denote a **quasi projective** scheme over a noetherian affine scheme $\mathrm{Spec}\,(A)$.
  You can assume $X = \mathrm{Spec}\,(A)$.
- We use the usual notations

$$\begin{cases} QCoh(X) = Category\ of\ quasi\ coherent\ modules\ on\ X \\ Coh(X) = Category\ of\ coherent\ modules\ on\ X \\ \mathscr{V}(X) = Category\ of\ locally\ free\ modules\ on\ X \end{cases}$$

Perfect Modules on X

- Recall, $grade(M) = \min\{r : \mathcal{E}xt^r(M, \mathcal{O}_X) \neq 0\}$.
  If X is Cohen-Macaulay then

$$grade(M) = co\dim(\mathrm{Supp}(M)) = height(ann(M))$$

- For a closed subset $Z \subseteq X$,

$$grade(Z, X) := grade(\mathcal{O}_Z)$$

with any subscheme structure $\mathcal{O}_Z$ on Z.

Perfect Modules

- A module $M \in Coh(X)$ will be called a **Perfect-module**

  $$\textit{if} \qquad grade(M) = \dim_{\mathscr{V}(X)} M$$

- They have only one non vanishing $\mathcal{E}xt^{k}(M, \mathcal{O}_X) \neq 0$.
- **Perfect** modules will also be referred to as $C\mathbb{M}$-modules on X, or $C\mathbb{M}(X)$-modules. There is a limited amount of literature on these.

Examples

- Suppose $\mathscr{I} \subseteq \mathcal{O}_X$ is a locally complete intersection ideal with $height(\mathscr{I}) = k$. Then

$$M = \frac{\mathcal{O}_X}{\mathscr{I}} \quad \text{is a } C\mathbb{M}(X)\text{−module.}$$

Notations

Let $Z \subseteq X$ be closed. Denote the full subcategories

$$\begin{cases} Coh^Z(X) = \{M \in Coh(X) : \mathrm{Supp}\,(M) \subseteq Z\} \\ If\ \ Z = V(I), M \in Coh^Z(X) \Longleftrightarrow V(ann(M)) \subseteq V(I) \\ \mathbb{M}(X) = \{M \in Coh(X) : \dim_{\mathscr{V}(X)}(M) < \infty\} \\ \mathbb{M}^Z(X) = \{M \in \mathbb{M}(X) : \mathrm{Supp}\,(M) \subseteq Z\} \\ C\mathbb{M}^Z(X) = \left\{M \in \mathbb{M}^Z(X) : \dim_{\mathscr{V}(X)}(M) = grade\,(\mathcal{O}_Z)\right\} \\ M \in C\mathbb{M}^Z(X) \Longrightarrow M \text{ is perfect}. \\ \mathrm{D}^b(\mathscr{E}) = \text{Bounded Derived Category of } \mathscr{E} \end{cases}$$

Derived Equivalence Theorem

The following functors are **equivalences of derived categories**.

$$\mathrm{D}^b(C\mathbb{M}^Z(X)) \xrightarrow[\zeta]{\sim} \mathrm{D}^b(\mathbb{M}^Z(X)) \xrightarrow[\iota]{\sim} \mathscr{D}^Z(\mathbb{M}(X)) \xleftarrow[\iota']{\sim} \mathscr{D}^Z(\mathscr{V}(X))$$

Here

$$\begin{cases} \mathscr{D}^Z(\mathscr{V}(X)) = \textit{Bounded Derived category of complexes} \\ \textit{whose homologies have support in } Z \\ \mathscr{D}^Z(\mathbb{M}(X)) = \cdots \end{cases}$$

K-Theory and Commutative Algebra

- K-Theory is an **invariant of such** Derived categories.
- Main ingredient to prove these Derived equivalences comes from the some improvisation of some commutative algebra methods, as follows.

Approximation Theorem

X be quasi projective, $Z \subseteq X$ closed, $grade(Z, X) = k$. Let

$$\cdots \longrightarrow \mathscr{F}_k \longrightarrow \cdots \longrightarrow \mathscr{F}_1 \longrightarrow \mathscr{F}_0 \longrightarrow \mathscr{F}_{-1}$$

be a complex in $Coh(X) \ni \mathcal{H}_i(\mathscr{F}_\bullet) \in Coh^Z(X)$, $\forall 0 \leq i \leq k$. Then there are maps of complexes $\nu_\bullet : \mathscr{E}_\bullet \longrightarrow \mathscr{F}_\bullet$ as follows:

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & \mathscr{E}_k & \longrightarrow & \cdots & \longrightarrow & \mathscr{E}_1 & \longrightarrow & \mathscr{E}_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \nu_k} & & & & \downarrow{\scriptstyle \nu_1} & & \downarrow{\scriptstyle \nu_0} & & \\ \cdots & \longrightarrow & \mathscr{F}_k & \longrightarrow & \cdots & \longrightarrow & \mathscr{F}_1 & \longrightarrow & \mathscr{F}_0 & \longrightarrow & \mathscr{F}_{-1} \end{array}$$

Continued: Theorem:

such that

1. $\mathscr{E}_i \in \mathscr{V}(X)\ \forall i$, and $\mathscr{E}_i = 0$ unless $0 \leq i \leq k$.
2. The map $\mathcal{H}_0(\nu_\bullet) : \mathcal{H}_0(\mathscr{E}_\bullet) \twoheadrightarrow \mathcal{H}_0(\mathscr{F}_\bullet)$ is surjective.
3. $\mathcal{H}_0(\mathscr{E}_\bullet) \in C\mathbb{M}^k(X)$ and $\mathcal{H}_i(\mathscr{E}_\bullet) = 0\ \forall i \neq 0$.

Commutative Algebra: Comments:

- $\mathscr{E}_\bullet$ would be a direct sum of certain Koszul complexes.
- Method of proofs goes back to a unpublished paper of Hans-Bjørn Foxby, executed for finite length homologies. Same was used in the paper of Roberts-Srinivas.
- In our case, we embed $X \subseteq Y = \mathrm{Proj}\,(S)$. The proof essentially reduces to the case, when $X = Y = \mathrm{Proj}\,(S)$. Then the method of **Prime avoidance works**.

K-Theory Spaces

$$\mathit{Applications}\ \ \mathit{To}\ \ \mathrm{K}-\mathit{Theory}$$

K-Theory Spaces

- Define the *K-Theory space* of a (small) exact category $\mathscr{E}$,

$$\mathrm{K}(\mathscr{E}) = \Omega\,|N_\bullet(\mathbb{Q}\mathscr{E})|$$

$$\begin{cases} \mathbb{Q}\mathscr{E} = \textit{the } \mathbb{Q} \textit{ category of } \mathscr{E} \\ N_\bullet(\mathbb{Q}\mathscr{E}) = \text{the Nerve of } \mathbb{Q}\mathscr{E}; \text{ a simplicial set} \\ |N_\bullet(\mathbb{Q}\mathscr{E})| = \textit{the Geometric realization}, \\ \text{AKA the classifying space of } \mathbb{Q}\mathscr{E} \\ \Omega\,|N_\bullet(\mathbb{Q}\mathscr{E})| = \text{the loop space } (\textit{space of all loops}) \end{cases}$$

K-Groups

- The K-groups are defined as $\mathrm{K}_n(\mathscr{E}) = \pi_n(\mathrm{K}(X))\ \forall n \geq 0$
- The K-theory spectra is also defined, subsequently by Waldhausen and Thomason to obtain negative K-groups, which we postpone or skip.

As Functors

- Let CatExact denote the category of (small) exact categories, and Exact Functors.
- The association

$$\begin{cases} \mathscr{E} \mapsto \mathrm{K}(\mathscr{E}) \\ F \mapsto \mathrm{K}(F) \end{cases} \quad \begin{cases} \text{is a functor} \quad \underline{\text{CatExact}} \longrightarrow \underline{\text{Top}} \\ \textit{In fact, } \mathrm{K}(\mathscr{E}) \textit{ is a CW complex.} \end{cases}$$

- The association

$$\begin{cases} \mathscr{E} \mapsto \mathrm{K}_n(\mathscr{E}) \\ F \mapsto \mathrm{K}_n(F) \end{cases} \quad \begin{cases} \text{is a sequence of functors} \\ \underline{\text{CatExact}} \longrightarrow \underline{\text{Ab}} \end{cases}$$

Primary Interest

My primary interest has been in the K-Theory of schemes.

I work on quasi projective schmens X over $\mathrm{Spec}\,(A)$.

$Z \subseteq X$ be closed, $U = X - Z$.

- The K-Theory of $Coh(X)$ and $\mathscr{V}(X)$ are of interest.
- Relationships between the K-theories,
  $\mathrm{K}\,(Coh(X))$, $\mathrm{K}\,(Coh(U))$, $\mathrm{K}\,(Coh(Z))$, and
  $\mathrm{K}\,(\mathscr{V}(X))$, $\mathrm{K}\,(\mathscr{V}(U))$, $\mathrm{K}\,(\mathscr{V}(Z))$ are of interest.
  Clarifications are needed, what kind of
  scheme structure we impose on Z, if any.

Primary Interest

- Quillen's results on $\mathrm{K}\left(Coh(X)\right)$, are most up to date. So, rest of the talk, we focus on $\mathrm{K}\left(\mathscr{V}(X)\right)$.
- There is also interest in **Grothendieck-Witt theory**, by incorporating dualities, when available. For example, there are natural dualities

$$\begin{cases} On\ \mathscr{V}(X) & \mathcal{E} \mapsto \mathcal{H}om\left(\mathcal{E}, \mathcal{O}_X\right) \\ On\ \mathrm{C}\mathbb{M}^Z(X) & M \mapsto \mathcal{E}xt^{grade(Z)}\left(M, \mathcal{O}_X\right) \end{cases}$$

Theory works fairly similar to K-Theory. I only outline the results.

Primary Interest

- It follows from the resolution theorem that

$$\mathsf{K}(\mathscr{V}(X)) \xrightarrow{\sim} \mathsf{K}(\mathbb{M}(X))$$

is a homotopy equivalence. Consequently,

$$\mathsf{K}_n(\mathscr{V}(X)) \xrightarrow{\sim} \mathsf{K}_n(\mathbb{M}(X)) \quad \text{is isomorphism, } \forall\ n \geq 0.$$

Exercise: Prove $K_0(\mathscr{V}(X)) \cong K_0(\mathbb{M}(X))$.

- So, our focus is now on $\mathbb{M}(X)$.

The Homotopy Fiber

As before X is quasi projective, $Z \subseteq X$ is closed, $U = X - Z$.

- Note U has a natural subscheme structure. Consider the map of K-Theory spaces $\mathrm{K}(\mathscr{V}(X)) \xrightarrow{\varepsilon} \mathrm{K}(\mathscr{V}(U))$
- Topologically, there is a Homotopy Fiber:

$$\mathscr{F}(\varepsilon) \overset{\iota}{\dashrightarrow} \mathrm{K}(\mathscr{V}(X)) \xrightarrow{\varepsilon} \mathrm{K}(\mathscr{V}(U))$$

The Triangle Fiber

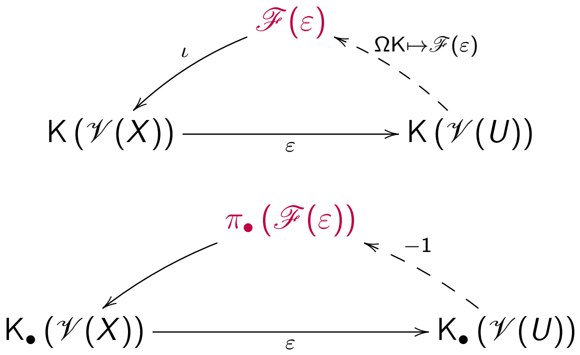
- This leads to a long exact sequence of Homotopy groups

$$\cdots \longrightarrow \pi_n(\mathscr{F}(\varepsilon)) \longrightarrow \mathrm{K}_n(\mathscr{V}(X)) \xrightarrow{\varepsilon} \mathrm{K}_n(\mathscr{V}(U))$$

$$\longrightarrow \pi_{n-1}(\mathscr{F}(\varepsilon)) \longrightarrow \cdots$$

The Triangles

- They are like Triangles of spaces:



The Description of K-theory fiber

- So, the question is what kind of Algebraic description can we give for $\mathscr{F}(\varepsilon)$, which should be basically depend on Z.
- **Main Result.** We have

$$\begin{array}{ll} \mathsf{K}\left(C\mathbb{M}^Z(X)\right) \stackrel{\sim}{\longrightarrow} \mathscr{F}(\varepsilon) & \text{is a homotopy equivalence} \\ \mathsf{K}_n\left(C\mathbb{M}^Z(X)\right) \stackrel{\sim}{\longrightarrow} \pi_n\left(\mathscr{F}(\varepsilon)\right) & \text{is an isomorphism } \forall \text{n} \end{array}$$

The Exact sequence

In fact, there is a long exact sequence of (negative) $\mathbb{K}$-groups

$$\begin{array}{ccccccc} \cdots & & \cdots & & \cdots & & \\ \mathrm{K}_1\left(C\mathbb{M}^Z(X)\right) & \longrightarrow & \mathrm{K}_1\left(\mathscr{V}(X)\right) & \xrightarrow{\varepsilon} & \mathrm{K}_1\left(\mathscr{V}(U)\right) & \longrightarrow & \\ \mathrm{K}_0\left(C\mathbb{M}^Z(X)\right) & \longrightarrow & \mathrm{K}_0\left(\mathscr{V}(X)\right) & \xrightarrow{\varepsilon} & \mathrm{K}_0\left(\mathscr{V}(U)\right) & \longrightarrow & \\ \mathbb{K}_{-1}\left(C\mathbb{M}^Z(X)\right) & \longrightarrow & \mathbb{K}_{-1}\left(\mathscr{V}(X)\right) & \xrightarrow{\varepsilon} & \mathbb{K}_{-1}\left(\mathscr{V}(U)\right) & \longrightarrow & \\ \cdots & & \cdots & & \cdots & & \end{array}$$

The Description of $\mathbb{G}W$-theory fiber

We prove exactly similar results in $\mathbb{G}W$-Theory.

Original Theorem of Quillen

The above directly extends the follow theorem of Quilllen (from the a paper of Daniel Grayson):

Theorem:(Quillen) Suppose X is a quasi compact scheme, $U = \mathrm{Spec}\,(A) \subseteq X$ is an affine open subschemes of X, and $Z = X - U$. Assume that Z has a subscheme structure, defined by an invertible ideal sheaf $\mathscr{I} \subseteq \mathcal{O}_X$. (*It means Z is local complete intersection of codimension one!*)

Continued

Consider the full subcategory:

$$\mathscr{H} = \left\{ \mathscr{F} \in QCoh(X) : \mathscr{F}_{|U} = 0, \dim_{\mathscr{V}(X)}(\mathscr{F}) = 1 \right\}$$

Then there is an exact sequence

$$\cdots \longrightarrow \mathrm{K}_n(\mathscr{H}) \longrightarrow \mathrm{K}_n(\mathscr{V}(X)) \longrightarrow \mathrm{K}_n(\mathscr{V}(U))$$

$$\longrightarrow \mathrm{K}_{n-1}(\mathscr{H}) \longrightarrow \cdots$$

ending at degree zero.

perspective

- A close look reveals $\mathscr{H} = C\mathbb{M}^Z(X)$. Thus my result above is exactly in tune with the result of Quillen, which works only when $co\dim(Z) = 1$. The result was written in a paper of Daniel Grayson.
- In between, a description of the homotopy fiber $\mathscr{F}(\varepsilon)$ was given by Thomason. This was stated in terms of K-Theory of the respective **chain complex category**.

Jargon of Spectra

To define negative homotopy groups, spectrum is defined in topology.

Definitions: A **Spectrum** is a sequence $\mathrm{E} = \{E_0, E_1, E_2, \cdots\}$, together with bonding maps $\sigma_n : E_n \longrightarrow \Omega E_{n+1}$. Here ΩE_{n+1} denote the loop space.

For such a spectrum and $n \in \mathbb{Z}$, define homotopy groups

$$\pi_n(\mathrm{E}) = \lim \left\{ \pi_{n+l}\left(\Omega^{k-l} E_k\right) : k \geq 0, n+l \geq 0, k-l \geq 0 \right\}$$

Jargon of Spectra

Given an exact category $\mathscr{E}$ we can define a spectrum $\mathbb{K}(\mathscr{E}) = \{\mathbb{K}^k(\mathscr{E}) : k \geq 0\}$. Consequently, $\mathbb{K}_n(\mathscr{E}) = \pi_n(\mathbb{K}(\mathscr{E}))$ is defined $\forall n \in \mathbb{Z}$. Our results above can be stated in terms of $\mathbb{K}$-theory spectrum.

Results on $\mathbb{K}$-theory Spetra

- There is a zig-zag sequence

$$\mathbb{K}\left(C\mathbb{M}^{Z}(X)\right) \longrightarrow \mathbb{K}\left(\mathscr{V}(X)\right) \longrightarrow \mathbb{K}\left(\mathscr{V}(U)\right)$$

  that is a homotopy fibration of $\mathbb{K}$-Theory spectra.
- We prove similar results on Grothedieck Witt theory Bi-specta.

Two Step Proof:

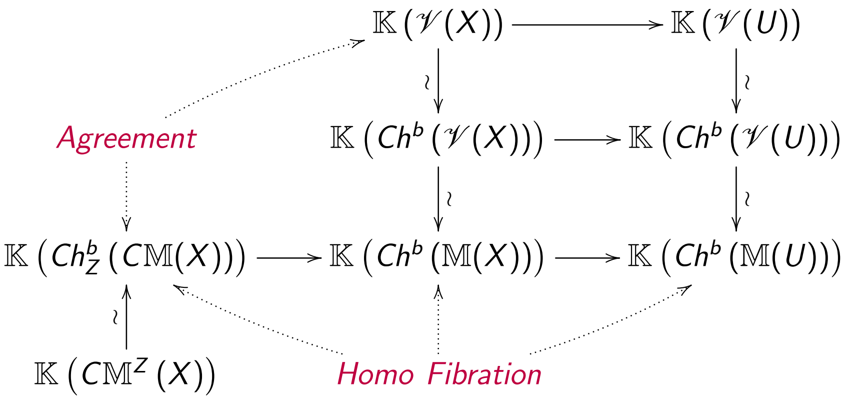
We check:

- There is an equivalence $\mathrm{D}^b\left(C\mathbb{M}^Z(X)\right) \xrightarrow{\sim} \mathscr{D}^Z\left(\mathscr{V}(X)\right)$.
- The zig-zag sequence

$$\begin{array}{ccccc}
\mathrm{D}^b\left(C\mathbb{M}^Z(X)\right) & & & & \\
\downarrow \wr & \searrow & & & \\
\mathscr{D}^Z\left(\mathscr{V}(X)\right) & \longrightarrow & \mathrm{D}^b\left(\mathscr{V}(X)\right) & \longrightarrow & \mathrm{D}^b\left(\mathscr{V}(U)\right)
\end{array}$$

is exact up to direct summand (*up to factor*).

Two Step Proof:



Routine Applications

There are some routine applications of the exact sequences, as above. We mention two.

The Singular Locus

Corollary: Let X be a quasi projective scheme over $\mathrm{Spec}\,(A)$ and Z be the singular locus of X and $U = X - Z$. Then we have an exact sequence

$$\cdots \longrightarrow K_1(\mathscr{V}(X)) \longrightarrow K_1(\mathscr{V}(U)) \longrightarrow$$

$$K_0(C\mathbb{M}^Z(X)) \longrightarrow K_0(\mathscr{V}(X)) \longrightarrow K_0(\mathscr{V}(U)) \longrightarrow$$

$$\mathbb{K}_{-1}(C\mathbb{M}^Z(X)) \longrightarrow \mathbb{K}_{-1}(\mathscr{V}(X)) \longrightarrow 0$$

The Singular Locus

Further,

$$\mathbb{K}_{-n}(C\mathbb{M}^Z(X)) \xrightarrow{\sim} \mathbb{K}_{-n}(\mathscr{V}(X)) \textit{ is isomorphism } \quad \forall n \geq 2$$

Proof. Note that U is non singular. So, we have $\mathbb{K}_n(\mathscr{V}(U)) = 0 \ \forall n \leq -1$. ■

A point support

Corollary: Let X be a quasi projective scheme over $\mathrm{Spec}\,(A)$ and $Z = V(\mathfrak{m})$ be a closed point in X and $U = X - Z$ be punctured open schem. Then we have an exact sequence

$$\cdots \longrightarrow K_1(\mathscr{V}(X)) \longrightarrow K_1(\mathscr{V}(U)) \longrightarrow$$

$$K_0(C\mathbb{M}^{V(\mathfrak{m})}(X)) \longrightarrow K_0(\mathscr{V}(X)) \longrightarrow K_0(\mathscr{V}(U)) \longrightarrow$$

$$\mathbb{K}_{-1}(C\mathbb{M}^{V(\mathfrak{m})}(X)) \longrightarrow \mathbb{K}_{-1}(\mathscr{V}(X)) \longrightarrow \cdots$$

Continued

If the residue field $\kappa(\mathfrak{m}) \in \mathbb{M}(X)$ then a lot is known about $\mathbb{K}_n\left(C\mathbb{M}^{V(\mathfrak{m})}(X)\right)$.

Notations

Let $k \geq 0$ be an integer. Denote

$$\begin{cases} \mathbb{M}(X) = \{M \in Coh(X) : \dim_{\mathscr{V}(X)}(M) < \infty\} \\ \mathbb{M}^k(X) = \{M \in \mathbb{M}(X) : grade(M) \geq k\} \\ C\mathbb{M}^k(X) = \left\{M \in \mathbb{M}(X) : grade(M) = \dim_{\mathscr{V}(X)}(M) = k\right\} \\ X^{(k)} = \{\wp \in X : co\dim(V(\wp)) = k\} \\ \forall x \in X,\ X_x := \mathrm{Spec}\left(\mathcal{O}_{X,x}\right) \end{cases}$$

Older results: Gersten complexes

Using same kind of methods, we have older results:

- Assume X is Cohen Macaulay. Then the sequence

$$\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}\left(C\mathbb{M}^{k}(X)\right) \longrightarrow \coprod_{x\in X^{(k)}} \mathbb{K}\left(C\mathbb{M}^{k}(X_x)\right)$$

  is a Homotopy Fibration of $\mathbb{K}$-Theory spectra.
- This leads up to the Gersten $\mathbb{K}$-Theory complexes.
- We have exactly similar results on Gersten $\mathbb{G}W$-Theory complexes.

Thank you!