# Splitting property of projective modules, by Homotopy obstructions

Satya Mandal
*Department of Mathematics, KU*

14 March 2019

# Prelude

The Theory for vector bundles in topology shaped the research in projective modules in algebra, consistently. This includes Obstruction Theory. The algebra has always been trying to catch up. To an extent, this fact remained under appreciated.

*I would avoid talking about such topological background, unless there is interest.*

# Search for a newer direction

After the conjecture of Serre was solved, a researchers sought newer directions, what to do next? .

# Chern Classes

Mohan Kumar and M. P. Murthy considered:
**Question:** Suppose $A$ is smooth affine algebra over an algebraically closed field $k$, with $\dim A = d$. Suppose $P$ is a projective $A-$module with $rank(P) = d$.

$$Does \quad C^d(P) = 0 \quad \Longrightarrow \quad P \approx Q \oplus A?$$

Here $C^d(P)$ denotes the top Chern class of $P$.

# Murthy's Theorem

## Theorem (Murthy)

*Suppose A is an affine algebra (smooth) over an algebraically closed field k, with* $\dim A = d$. *Let P be a projective* $A-$*module with* $rank(P) = d$.

$$Then \quad C^d(P) = 0 \quad \Longleftrightarrow \quad P \approx Q \oplus A$$

**Remark.** Similar obstruction classes $e(P)$, is a suitable obstruction set (preferably a group), for a wider class of rings $A$ and for any $rank(P) \leq d$, was sought.

# The Homotopy Program, for projective Modules

In search for an obstruction for, a projective $A$-module $P$, to split of a free direct summands, Nori provided two germs of ideas, apparently related.

- First one is so called Homotopy Question.
- For a commutative ring $A$ with $\dim A = d$, a definition of so called Euler class group $E^d(A, A)$. Also for a projective $A$-module $P$ with $rank(P) = d$ and $det(P) = A$, an Euler class $e(P) \in E^d(A, A)$ was defined. It was conjectured (and proved [?])

$$e(P) = 0 \Longleftrightarrow P \cong Q \oplus A$$

# The Homotopy Question

Suppose $A$ is a noetherian commutative ring with $\dim A = d$. Let $P$ be a Projective $A$-module, with $rank(P) = n$.

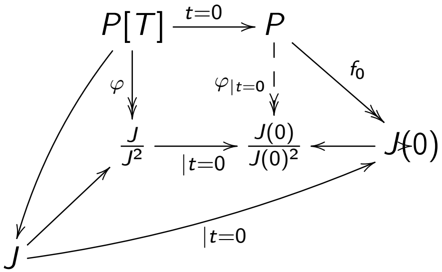
- Let $A[T]$ be the polynomial ring, and $P[T] := P \otimes A[T]$.
- Let $\varphi : P[T] \twoheadrightarrow \frac{J}{J^2}$ and $f_0 : P \twoheadrightarrow J(0)$ be surjective maps.
- Assume $\varphi$ and $f_0$ are compatible, meaning the diagram

$$\begin{array}{ccccc} P[T] & \xrightarrow{t=0} & P & & \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_{|t=0}} & \searrow^{f_0} & \\ \frac{J}{J^2} & \xrightarrow[|t=0]{} & \frac{J(0)}{J(0)^2} & \longleftarrow & J(0) \end{array}$$

commute

# Continued

Can we extend the above diagram, as follows?:




commute

# Continued

Without diagram, the question is does there exist a surjective map

$$\Psi : P[T] \twoheadrightarrow J \quad \ni \quad \Psi_{|t=0} = f_0$$

And, $\Psi$ is a lift of $\varphi$, meaning

$$\begin{array}{ccc} P[T] & \overset{\Psi}{\twoheadrightarrow} & J \\ & \searrow^{\varphi} & \downarrow \\ & & \frac{J}{J^2} \end{array} \qquad \text{commute}$$

# Remark

Point is, if the Homotopy Problem has a solution then

$$\Phi_{|t=1} : P \twoheadrightarrow J(1) \quad \textit{is a surjective lift of} \quad \varphi_{|t=1} : P \twoheadrightarrow \frac{J(1)}{J(1)^2}$$

In other words, if $\varphi_{|t=0}$ "Good", then so would be $\varphi_{|t=1}$!

# Topological Theorem of Nori

The above question is exact translation of the following theorem of Nori.

**Theorem** Suppose $M$ is a manifold, and $V$ is a real vector bundle over $M$, with $rank(V) = n$. Let $B \subseteq M \times \mathbb{R}$ be a smooth submanifold of $M \times \mathbb{R}$ with $co\dim B = n$. Let

$$\begin{cases} s \in \Gamma(M, V) \text{ and } B_0 = \{s = 0\}, \ s \perp 0 (\text{i.e transversal}) \\ \varphi : N(M \times \mathbb{R}, B) \xrightarrow{\sim} (p^* V)_{|B} \quad \text{be an isomorphism} \\ \ni \ \ B \cap M \times 0 = B_0 \times 0, \ [s] : N(M, B_0) \xrightarrow{\sim} V_{|B_0} \end{cases}$$

That means, $\varphi$ and $s$ are "compatible".

# Continued

If $2n \geq \dim M + 3$ then, there is a section $\Psi \in \Gamma(M \times \mathbb{R}, p^*V)$, such that

$$\Psi_{t=0} = s, \qquad \{\Psi = 0\} = B \text{ and } \qquad \Psi_{|B} = \varphi$$

# Bhatwadekar-Keshari's Result

Best Affirmative answer to Homotopy Question is due to S. M. Bhatwadekar and Manoj Keshari, under the hypotheses

- $A$ is smooth over and infinite perfect field $k$.
- $2n \geq d + 3$, and $height(I) \geq n$.

We use this result!

# Local Orientations

A commutative noetherian ring is denoted by $A$, with $1/2 \in A$. $P$ denote a projective $A$-module, and $d = \dim A$, $rank(P) = n$.

**Definition:** A **Local $P$-orientation**, is a pair $(I, \omega)$, such that

- $I$ is an ideal of $A$,
- $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is a surjective homomorphism.

**Main Problem:** Whether $\omega$ lifts to a surjective map $P \twoheadrightarrow I$? Whether this can be detected by means of homotopy obstruction properties?

# The Homotopy Obstruction Set

- Let $\mathcal{L}O(P) = \text{Set of all local P} - \text{orientations}$
- The maps $\mathcal{L}O(P) \xleftarrow{T=0} \mathcal{L}O(P \otimes A[T]) \xrightarrow{T=1} \mathcal{L}O(P)$ induce a chain homotopy equivalence on $\mathcal{L}O(P)$.
- **Definition:** The homotopy obstruction set $\pi_0(\mathcal{L}O(P))$ is defined to be the set of all equivalence classes.
- For $(I, \omega) \in \mathcal{L}O(P)$, its image in $\pi_0(\mathcal{L}O(P))$,

$$\text{is denoted by} \quad [(I, \omega)] \in \pi_0(\mathcal{L}O(P))$$

# Obstruction Class

- There are two distinguished elements in $\mathcal{LO}(P)$, namely, $(\mathbf{0}, 0)$ and $(A, 0)$. Denote,

$$\left\{ \begin{array}{l} \mathbf{e}_0 = [(\mathbf{0}, 0)] \in \pi_0\left(\mathcal{LO}(P)\right) \\ \mathbf{e}_1 = [(A, 0)] \in \pi_0\left(\mathcal{LO}(P)\right) \\ \text{Define} \quad \varepsilon(P) = \mathbf{e}_0 \in \pi_0\left(\mathcal{LO}(P)\right) \end{array} \right.$$

  to be called, the **(Nori) Homotopy Class**, of $P$.
- In fact, $\varepsilon(P) = \mathbf{e}_0 = [(I, \overline{f})]$, induced by any surjective map $f : P \twoheadrightarrow I$.

# The Equivalence Theorem

**Theorem E** Suppose $A$ is a regular ring, containing a field $k$, with $1/2 \in k$, and $\dim A = d \geq 2$. Let $P$ be a projective $A$-module with $ramk(P) = n$. Then, the chain equivalence relation on $\mathcal{LO}(P)$, is an equivalence relation.

**Proof.** Use the quadratic structure on $P^* \oplus P \oplus A$. If $P = A^n$ is free, the quadratic structure is:

$$\sum_{i=1}^{n} X_i Y_i + Z^2$$

# The Splitting Theorem

**Theorem Split:** Suppose $A$ is (essentially) smooth, over an infinite perfect field $k$, with $1/2 \in k$, and $\dim A = d \geq 2$. Let $P$ be a projective $A$-module with $ramk(P) = n$. Assume $2n \geq d + 3$. Then,

$$P \cong Q \oplus A \quad \Longleftrightarrow \quad \varepsilon(P) = \mathbf{e}_1 \qquad \text{(the additive zero)}$$

# The Lifting Theorem

**Theorem Lift:** Suppose $A$ is (essentially) smooth, over an infinite perfect field $k$, with $1/2 \in k$, and $\dim A = d \geq 2$. Let $P$ be a projective $A$-module with $ramk(P) = n$. Assume $2n \geq d + 3$. Then,
For local orientations $(I, \omega)$, with $height(I) \geq n$, then $\omega$ lifts to surjective map:

$$\begin{array}{ccc} P & \overset{\Omega}{\dashrightarrow\!\!\!\!\rightarrow} & I \\ & \searrow^{\omega} & \downarrow \\ & & \frac{I}{I^2} \end{array} \quad \Longleftrightarrow \quad \varepsilon(P) = [(I, \omega)] \in \pi_0(\mathcal{LO}(P))$$

# The Monoid Theorem

**Theorem Monoid** Suppose $A$ is a regular ring, containing a field $k$, with $1/2 \in k$, and $\dim A = d \geq 2$. Let $P$ be a projective $A$-module with $ramk(P) = n$. Assume $2n \geq d + 2$. Then, the obstruction set $\pi_0(\mathcal{L}O(P))$ has an additive structure.

- Addition is associative and commutative.
- $\mathbf{e}_1$ is the identity.

Also, if $\varepsilon(P) = \mathbf{e}_1$, then $\pi_0(\mathcal{L}O(P))$ is a group. This would be the case, if $P \cong Q \oplus A$.

# The Comparisons

We would compare $\mathcal{LO}(P)$ with Chow groups, and so called Euler Class groups.

For each projective $A$-module $P$, we would define a **Euler Class Group** $E(P)$. If $P = A^n$, then $E(P) = E^n(A, A)$ defined by Nori and others.

# Definition of $E(P)$

As before, assume $rank(P) = n$. Let

$$\begin{cases} \mathcal{LO}^n(P) = \{(I, \omega_I) \in \mathcal{LO}(P) : height(I) = n\}, \\ \mathcal{LO}^n_c(P) = \{(I, \omega_I) \in \mathcal{LO}(P)^n : V(I) \text{ is connected}\}. \\ \text{Define} \quad E(P) = \frac{\mathbb{Z}(\mathcal{LO}^n_c(P))}{\mathscr{R}(P)} \end{cases}$$

where $\mathscr{R}(P)$ is the subgroup, generated by global orientations.

# Comparison with $E(P)$

**Theorem EP:** Suppose $A$ is a regular ring, containing a field $k$, with $1/2 \in k$, and $\dim A = d \geq 2$. Assume $rank(P) = n$ and $2n \geq d+2$. Assume $P = Q \oplus A$, then there is a surjective homomorphism

$$\varphi : E(P) \twoheadrightarrow \pi_0\left(\mathcal{LO}(P)\right)$$

If $A$ is essentially smooth over a perfect field, and $2n \geq d+3$, then $\varphi$ is an isomorphism.

# Chow Group: Case $k = \overline{k}$

We exploit the work of N.Mohan Kumar and M. P.Murthy.

**Theorem Chow-1:** Suppose is a $A$ is a smooth affine algebra over an algebraically closed a field $k$, with $1/2 \in k$, and $\dim A = d \geq 3$. Assume $rank(P) = d$. Then the following are isomorphisms

$$\varphi : E(P) \xrightarrow{\sim} \pi_0\left(\mathcal{LO}(P)\right) \xrightarrow{\sim} CH^d(A)$$

where $CH^d(A)$ denote the Chow Group of zero cycles.

# Chow Group: General Case

**Theorem Chow-2:** Suppose $A$ is a regular ring, containing a field $k$, with $1/2 \in k$, $\dim A = d$. Let $P$ be a projective $A$-module, with $rank(P) = n$. Then, there is a natural set theoretic map

$$\pi_0\left(\mathcal{L}O(P)\right) \longrightarrow CH^n(A) \quad \text{sending} \quad (I, \omega) \mapsto cycle(I)$$

In particular,

$$\varepsilon(P) \mapsto C^n(P^*)$$

# Top Rank Case

**Theorem:** Suppose $A$ is a regular ring, containing a field $k$, with $1/2 \in k$, $\dim A = d$. Let $P$, $Q$ be projective $A$-modules, with $rank(P) = rank(Q) = d$. Let $\iota : \Lambda^d Q \xrightarrow{\sim} Lambda^d P$ be an isomorphism. Then, the following are isomorphisms:

$$\chi(\iota) : \pi_0(\mathcal{LO}(Q)) \xrightarrow{\sim} \pi_0(\mathcal{LO}(P))$$

- So, $\pi_0(\mathcal{LO}(P))$ is a group, even if $P$ does not split.