Homotopy obstructions for projective modules

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Prelude

The Theory for vector bundles in topology shaped the research in projective modules in algebra, consistently. This includes Obstruction Theory. The algebra has always been trying to catch up. To an extent, this fact remained under appreciated.

Rings

- A ring A is a set with an addition $(+)$ and a multiplication. It is a commutative group under addition $+$, and the multiplication is distributive with respect to $+$.

  So, $\mathbb{Z}, \mathbb{R}, \mathbb{C}$ are rings.

- Let $\mathcal{X}$ be a topological space. Let

$$C(\mathcal{X}) = \{f : \mathcal{X} \longrightarrow \mathbb{R} : f \text{ is continuous.}\}$$

  Then, $C(\mathcal{X})$ is a ring.

Modules

- A module M over a ring A is, what a vector space would be over a field.
- A free module F over a ring A is an $A-$module that has a basis. If F is a finitely generated free $A-$module, then $F \approx A^n$. In this case, define $rank(F) := n$.

Vector bundles

Suppose $\mathcal{X}$ is a topological space. A (real) vector bundle on $\mathcal{X}$, is a continuous map $p : \mathcal{E} \to \mathcal{X}$ such that

- Each fiber $\mathcal{E}_x = p^{-1}(x)$ has a vector space structure.
- $\mathcal{X}$ has an open cover $\{U_i\}$ and homeomorphisms (trivializations) φ_i such that the diagrams

$$\begin{array}{ccc} p^{-1}(U_i) & \xrightarrow[\sim]{\varphi_i} & U_i \times \mathbb{R}^r \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & U_i & \end{array}$$

commute.

- For each $x \in U_i$, the trivialization φ_i induces linear isomorphisms $\mathcal{E}_x \to \mathbb{R}^r$.

Vector bundles

- The rank of $\mathcal{E}$ is defined as $rank(\mathcal{E}) = r$.
- Example: $\mathcal{X} \times \mathbb{R}^r \to \mathcal{X}$ *is the trivial bundle on* $\mathcal{X}$.
- Example: *The tangent bundle* $\mathcal{T}$ *over a manifold* $\mathcal{X}$, *is a vector bundle.*

The Module of Sections

Given a vector bundle $\mathcal{E} \longrightarrow \mathcal{X}$, as above, let

$$\Gamma(\mathcal{E}) := \{s : \mathcal{X} \to \mathcal{E} : ps = Id_{\mathcal{X}}, \quad s \textit{ is continuous}\}.$$

This means $s(x) \in \mathcal{E}_x \ \ \forall x \in \mathcal{X}$.

1. Elements $s \in \Gamma(\mathcal{E})$ are called sections of $\mathcal{E}$.
2. $\Gamma(\mathcal{E})$ is a $C(\mathcal{X})-$module. Today, I would call them the best examples of a rings and a modules. In fact, $\Gamma(\mathcal{E})$ is a "Projective" $C(\mathcal{X})$-module, to be defined next.

Projective Modules

Suppose A is a commutative ring. An $A-$module P is said to be projective, if

$$P \oplus Q = Free$$

for some other $A-$module Q.

The Correspondence theorem of Swan

Theorem ([Swan 1962])

Suppose $\mathcal{X}$ is a (compact connected) Hausdorff topological space. The functor

$$\Gamma : \mathcal{V}(\mathcal{X}) \longrightarrow \mathcal{P}(C(\mathcal{X})) \quad \textit{sending} \quad \mathcal{E} \to \Gamma(\mathcal{E})$$

is an equivalence of categories, where

- *$\mathcal{V}(\mathcal{X})$ denotes the category of vector bundles over $\mathcal{X}$*
- *and $\mathcal{P}(C(\mathcal{X}))$ denotes the category of of finitely generated projective $C(\mathcal{X})-$modules.*

Noetherian Rings

- The ring $C(\mathcal{X})$ is too big. We work with the ring of algebraic functions only. Algebraic functions are those that are, roughly, the restrictions of polynomial function.
- I will often talk about "noetherian commutative rings," because the ring of algebraic functions over a space M are "notherian and commutative".

Never-Vanishing sections

- Let $\mathcal{X}$ be a real manifold with $\dim \mathcal{X} = d$.
- Let $\mathcal{E}$ be a vector bundle of rank r.
- If $r > d$, then $\mathcal{E}$ has a never-vanishing section. This translates to

$$\Gamma(\mathcal{E}) \approx Q \oplus C(\mathcal{X}) \quad as\ C(\mathcal{X}) - modules.$$

Splitting

The above inspired the theorem of Serre ([Serre1957]):

- Let A be a noetherian commutative ring with $\dim A = d$.
- Let P be a projective $A-$module of rank r.
- If $r > d$, then P has a free direct summand.

$$\textit{This means} \quad P \approx Q \oplus A.$$

Polynomial rings

- $\mathbb{R}^n$ is contractible. So, vector bundles over $\mathbb{R}^n$ are trivial.
- So, J.-P. Serre conjectured ([Serre1955]) the same for polynomial rings.
- Independently, Quillen and Suslin proved the conjecture:

Polynomial rings

Theorem ([Quillen1976], [Suslin1976])

Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k. Then, finitely generated projective $A-$modules P are free.

Search for a newer direction

After the conjecture of Serre was solved, a researchers sought newer directions, what to do next? .

Chern Classes

Mohan Kumar and M. P. Murthy considered:
Question: Suppose A is smooth affine algebra over an algebraically closed field k, with $\dim A = d$. Suppose P is a projective $A-$module with $rank(P) = d$.

$$Does \quad C^d(P) = 0 \quad \Longrightarrow \quad P \approx Q \oplus A?$$

Here $C^d(P)$ denotes the top Chern class of P.

Murthy's Theorem

Theorem (Murthy)

Suppose A is an affine algebra (smooth) over an algebraically closed field k, with $\dim A = d$. Let P be a projective $A-$module with $rank(P) = d$.

$$\textit{Then} \quad C^d(P) = 0 \quad \Longleftrightarrow \quad P \approx Q \oplus A$$

Remark. Similar obstruction classes $e(P)$, is a suitable obstruction set (preferably a group), for a wider class of rings A and for any $rank(P) \leq d$, was sought.

More Definitions

- For a commutative ring A, a subset $I \subseteq A$ is called an IDEAL, if I is closed under addition and $AI \subseteq I$.
- We consider, commutative noetherian rings A.
  - You may think about A as the ring of algebraic functions on a topological space $\mathrm{Spec}(A)$.
  - An ideal $I \subseteq A$ is to be thought of set of functions $f \in A$, which vanish on a subset $V(I) \subseteq \mathrm{Spec}(A)$.
  - Still better, for $k = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ or a field, think of

$$A = \frac{k[X_1, \ldots, X_n]}{I} = k[x_1, \ldots, x_n]$$

The Homotopy Program, for projective Modules

In search for an obstruction for, a projective A-module P, to split of a free direct summands, Nori provided two germs of ideas, apparently related.

- First one is so called Homotopy Question.
- For a commutative ring A with $\dim A = d$, a definition of so called Euler class group $E^d(A, A)$. Also for a projective A-module P with $rank(P) = d$ and $det(P) = A$, an Euler class $e(P) \in E^d(A, A)$ was defined. It was conjectured (and proved [BhatSri])

$$e(P) = 0 \iff P \cong Q \oplus A$$

The Homotopy Question

Suppose A is a noetherian commutative ring with $\dim A = d$. Let P be a Projective A-module, with $rank(P) = n$.

- Let $A[T]$ be the polynomial ring, and $P[T] := P \otimes A[T]$.
- Let $\varphi : P[T] \twoheadrightarrow \frac{J}{J^2}$ and $f_0 : P \twoheadrightarrow J(0)$ be surjective maps.
- Assume φ and f_0 are compatible, meaning the diagram

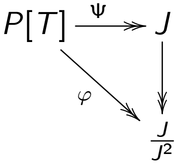
$$\begin{array}{ccccc} P[T] & \xrightarrow{t=0} & P & \xrightarrow{f_0} & J(0) \\ \downarrow \varphi & & \downarrow \varphi|_{t=0} & \swarrow & \\ \frac{J}{J^2} & \xrightarrow[|t=0]{} & \frac{J(0)}{J(0)^2} & & \end{array} \qquad \text{commute}$$

Continued

Question is does there exist a surjective map

$$\Psi : P[T] \twoheadrightarrow J \quad \ni \quad \Psi_{|t=0} = f_0$$

And, Ψ is a lift of φ, meaning



commute

Remark

Point is, if the Homotopy Problem has a solution then

$$\Phi_{|t=1} : P \twoheadrightarrow J(1) \quad \textit{is a surjective lift of} \quad \varphi_{|t=1} : P \twoheadrightarrow \frac{J(1)}{J(1)^2}$$

is a surjective lift of $\varphi_{|t=1} : P \twoheadrightarrow \frac{J(1)}{J(1)^2}$.
In other words, if $\varphi_{|t=0}$ "Good", then so is $\varphi_{|t=1}$.

Topological Theorem of Nori

The above question is exact translation of the following theorem of Nori.

Theorem Suppose M is a manifold, and V is a real vector bundle over M, with $rank(V) = n$. Let $B \subseteq M \times \mathbb{R}$ be a smooth submanifold of $M \times \mathbb{R}$ with $co\dim B = n$. Let

$$\left\{ \begin{array}{l} \varphi : N(M \times \mathbb{R}, B) \xrightarrow{\sim} (p^*V)_{|B} \quad \text{be an isomorphism} \\ s \in \Gamma(M, V) \text{ and } B_0 = \{s = 0\} \text{ is transversal} \\ \text{Let } [s] : N(M, B_0) \xrightarrow{\sim} V_{|B_0}, \ B \cap M \times 0 = B_0 \times 0 \end{array} \right.$$

Continued

Assume φ and s are compatible. If $2n \geq \dim M + 3$, the there is section $\Psi \in \Gamma(M \times \mathbb{R}, p^* V)$, such that

$$\Psi_{t=0} = s, \qquad \{\Psi = 0\} = B \quad \text{and} \qquad \Psi_{|B} = \varphi$$

Bhatwadekar-Keshari's Result

Best Affirmative answer to Homotopy Question is due to S. M. Bhatwadekar and Manoj Keshari, under the hypotheses

- A is smooth over and infinite perfect field k.
- $2n \geq d + 3$, and $height(I) \geq n$.

We use this result!

Local Orientations

A commutative noetherian ring is denoted by A, with $1/2 \in A$. P denote a projective A-module, and $d = \dim A$, $rank(P) = n$.

Definition: A **Local P-orientation**, is a pair (I, ω), such that

- I is an ideal of A,
- $\omega : P \twoheadrightarrow \frac{I}{I^2}$ is a surjective homomorphism.

Main Problem: Whether ω lifts to a surjective map $P \twoheadrightarrow I$? Whether this can be detected by means of homotopy obstruction properties?

The Homotopy Obstruction Set

- Let $\mathcal{LO}(P) = \text{Set of all local P} - \text{orientations}$
- The two maps $\mathcal{LO}(P) \xleftarrow{T=0} \mathcal{LO}(P \otimes A[T]) \xrightarrow{T=1} \mathcal{LO}(P)$ induce a chain homotopy equivalence on $\mathcal{LO}(P)$.
- **Definition:** The homotopy obstruction set $\pi_0(\mathcal{LO}(P))$ is defined to be the set of all equivalence classes.
- For a local P-orientation (I, ω) its image in $\pi_0(\mathcal{LO}(P))$,

$$\text{is denoted by} \quad [(I, \omega)] \in \pi_0(\mathcal{LO}(P))$$

Obstruction Class

- There are two distinguished elements in $\mathcal{LO}(P)$, namely, $(\mathbf{0},0)$ and $(A,0)$. Denote,

$$\begin{cases} \mathbf{e}_0 = [(\mathbf{0},0)] \in \pi_0\left(\mathcal{LO}(P)\right) \\ \mathbf{e}_1 = [(A,0)] \in \pi_0\left(\mathcal{LO}(P)\right) \end{cases}$$

- Define the **obstruction class**

$$\varepsilon(P) = \mathbf{e}_0 \in \pi_0\left(\mathcal{LO}(P)\right)$$

to be called, the **Nori Homotopy Class**, of P.

The Equivalence Theorem

Theorem E Suppose A is a regular ring, containing a field k, with $1/2 \in k$, and $\dim A = d \geq 2$. Let P be a projective A-module with $ramk(P) = n$. Then, the chain equivalence relation on $\mathcal{LO}(P)$, is an equivalence relation.

Proof. Use the quadratic structure on $P^* \oplus P \oplus A$. If $P = A^n$ is free, the quadratic structure is:

$$\sum_{i=1}^{n} X_i Y_i + Z^2$$

The Splitting Theorem

Theorem Split: Suppose A is (essentially) smooth, over an infinite perfect field k, with $1/2 \in k$, and $\dim A = d \geq 2$. Let P be a projective A-module with $ramk(P) = n$. Assume $2n \geq d + 3$. Then,

$$P \cong Q \oplus A \iff \varepsilon(P) = \mathbf{e}_1 \qquad \text{(the additive zero)}$$

The Lifting Theorem

Theorem Lift: Suppose A is (essentially) smooth, over an infinite perfect field k, with $1/2 \in k$, and $\dim A = d \geq 2$. Let P be a projective A-module with $ramk(P) = n$. Assume $2n \geq d+3$. Then,
For local orientations (I, ω), with $height(I) \geq n$, then ω lifts to surjective map:

$$\begin{array}{ccc} P & \overset{\Omega}{\dashrightarrow} & I \\ & {\scriptstyle \omega} \searrow & \downarrow \\ & & \frac{I}{I^2} \end{array} \qquad \Longleftrightarrow \qquad \varepsilon(P) = [(I, \omega)] \in \pi_0\left(\mathcal{LO}(P)\right)$$

The Monoid Theorem

Theorem Monoid Suppose A is a regular ring, containing a field k, with $1/2 \in k$, and $\dim A = d \geq 2$. Let P be a projective A-module with $ramk(P) = n$. Assume $2n \geq d + 2$. Then, the obstruction set $\pi_0(\mathcal{L}O(P))$ has an additive structure.

- Addition is associative and commutative.
- $\mathbf{e}_1$ is the identity.

Also, if $\varepsilon(P) = \mathbf{e}_1$, then $\pi_0(\mathcal{L}O(P))$ is a group. This would be the case, if $P \cong Q \oplus A$.

Comparison with Euler Class Groups

For each projective A-module P, we define a **Euler Class Group** $E(P)$, and compare it with $\pi_0(\mathcal{LO}(P))$. If $P = A^n$, then $E(P) = E^n(A, A)$ defined by others.

Theorem EP: Suppose A is a regular ring, containing a field k, with $1/2 \in k$, and $\dim A = d \geq 2$. Assume $rank(P) = n$ and $2n \geq d + 2$. Assume $P = Q \oplus A$, then there is a surjective homomorphism

$$\varphi : E(P) \twoheadrightarrow \pi_0(\mathcal{LO}(P))$$

If A is essentially smooth over a perfect field, and $2n \geq d + 3$, then φ is an isomorphism.

Comparison with Chow Groups

We exploit the work of N.Mohan Kumar and M. P.Murthy.

Theorem Chow Groups:
Suppose is a A is a smooth affine algebra over an algebraically closed a field k, with $1/2 \in k$, and $\dim A = d \geq 3$. Assume $rank(P) = d$. Assume $P = Q \oplus A$, then the following are isomorphisms

$$\varphi : E(P) \xrightarrow{\sim} \pi_0(\mathcal{L}O(P)) \xrightarrow{\sim} CH^d(A)$$

where $CH^d(A)$ denote the Chow Group of zero cycles.

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