Commutative Algebra and Algebraic K-Theory

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Abstract: Before the work of Quillen (1972), Higher Algebraic K-Theory was considered as a part of Commutative Algebra (Rings and Modules). In this talk we would discuss this author's recent efforts to bridge this artificial (tantalizing) gap between Commutative Algebra and Algebraic K-Theory, which developed during this last forty plus years. During the same period, Algebraic K-Theory also has progressed a long distance. Advent of negative $\mathbb{K}$-theory is among the greatest milestones.

For some further flavor, assume X is a quasi projective scheme. Given a chain complex map $\nu_\bullet : \mathcal{L}_\bullet \longrightarrow \mathcal{G}_\bullet$ between two complexes $\mathcal{L}_\bullet$, $\mathcal{G}_\bullet$, of coherent (or locally free) sheaves on X, one complex can be viewed as an approximation to the other. In general, constructing such approximations would be challenging. In the affine case $X = Spec(A)$, such a map was constructed by Hans-Bjørn Foxby (unpublished), using Koszul complexes. We implement this construction to quasi projective schemes. This can be considered as a "graded version" of Foxby's construction. The main point of this talk is, how we apply this approximating tool to (negative) $\mathbb{K}$-Theory and Grothendieck Witt ($\mathbb{G}W$)-Theory.

1 The Morphism Construction

First, consider affine case.

Lemma 1.1. Let $X = Spec(A)$ be an affine scheme. Consider the following diagram

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & & & & & & & & & & & \downarrow f & & \\ 0 & \longrightarrow & G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & N & \longrightarrow & 0 \end{array} \tag{1}$$

where $f : M \longrightarrow N$ is a homomorphim of A-modules, the first line is a projective resolution of M and the second line is an exact sequence of A-modules. Then, f extends to a map of complexes:

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow f_n & & \downarrow f_{n-1} & & & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\ 0 & \longrightarrow & G_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

Perhaps, the second line of (1) is the target of our interest, while the first line (or "some concoction") approximates the second line.

When the 2^{nd}-line is not exact, H.-B. Foxby [F, FH]:

Lemma 1.2.

Let $\quad G_\bullet : \quad 0 \longrightarrow G_n \longrightarrow G_{n-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow \cdots$

be a bounded complex of finitely generated A-modules.

Let $\quad f_1, f_2, \ldots, f_r \in \bigcap_{i \in \mathbb{Z}} \mathbf{Ann}\,(H_i(G_\bullet)) \quad$ be a sequence.

$\forall \; n \gg 0, \quad \mathcal{K}_\bullet := \mathcal{K}(f_1^n, f_2^n, \ldots, f_r^n) \quad$ denote the Koszul complex.

For $x \in H_0(G_\bullet)$, let $\zeta_x : H_0(\mathcal{K}_\bullet) \longrightarrow H_0(G_\bullet)$ denote the map $\overline{1} \mapsto x$. Then, there is a map of complexes

$$\varphi_\bullet^x : \mathcal{K}_\bullet \longrightarrow G_\bullet \qquad \ni \qquad H_0(\varphi_\bullet) = \zeta_x.$$

If $f_1, \ldots, f_n$ is regular then, $H_i(\mathcal{K}_\bullet) = 0 \; \forall \; i \neq 0$.

Taking direct sum of such Koszul complexes:

Theorem 1.3. Under the same set up as in (1.2): There is a bounded complex $P_\bullet$ of projective modules and a map $\varphi_\bullet : P_\bullet \longrightarrow G_\bullet$ of complexes, such that

1. $H_0(\varphi_\bullet) : H_0(P_\bullet) \twoheadrightarrow H_0(G_\bullet)$ is surjective.
2. In fact, $P_\bullet$ would be a direct sum of $\mathcal{K}_\bullet(f_1^n, \ldots, f_r^n)$, with $n \gg 0$.
3. Consequently, if $f_1, \ldots, f_r$ is regular, then $P_\bullet$ is a resolution of $H_0(P_\bullet)$ and

$$grade\,(H_0(P_\bullet)) = \operatorname{proj\,dim}(H_0(P_\bullet)) = r$$

Proof. Follows from (1.2). ■

Corollary 1.4. *Nothing prevents me from writing a graded version of the same (1.3) and sheaffify.*

Lemma 1.5. Let X be a quasi projective scheme over Spec(A). More precisely, $S = \oplus_{i \geq 0} S_i$ is a noetherian graded ring, with $S_0 = A$, $S = A[S_1]$ and Let X be an open subset of $\tilde{X} := Proj(S)$.

Let $Y \subseteq X$ be a closed subset of X, with

$$grade(\mathcal{O}_Y, \mathcal{O}_X) \geq r. \qquad \text{Write} \quad V(I) = \overline{Y}$$

be the closure of Y, where I is the homogeneous ideal of S, defining $\overline{Y}$. Then,

$$\exists \qquad f_1, \ldots, f_r \in I \qquad \ni \qquad f_{i_1}, \ldots, f_{i_j}$$

induce regular $S_{(\wp)}$-sequences on X.

Proof. Simple prime avoidance methods. ∎

Corollary 1.6. With $f_1, \ldots, f_r \in I$ in lemma 1.5, we have the following:

1. First, we can form the Koszul complexes $K_\bullet(f_1^n, \ldots, f_r^n)$ of graded modules.

2. By sheafification, we get the Koszul complexes $\mathcal{K}_\bullet(f_1^n, \ldots, f_r^n)$, which is in $Ch^b(Coh(\mathscr{V}(\mathcal{X}))) \hookrightarrow Ch^b(Coh(\mathcal{X}))$.

3. Its restriction $\mathcal{K}_\bullet(f_1^n, \ldots, f_r^n)_{|X}$, is in $Ch^b(Coh(\mathscr{V}(X))) \hookrightarrow Ch^b(Coh(X))$.

   In deed, $\mathcal{K}_\bullet(f_1^n, \ldots, f_r^n)_{|X}$ is a resolution of $\mathcal{H}_0\left(\mathcal{K}_\bullet(f_1^n, \ldots, f_r^n)_{|X}\right)$ and

   $$grade\left(\mathcal{H}_0\left(\mathcal{K}_\bullet(*)_{|X}\right)\right) = \dim_{\mathscr{V}(X)}\left(\mathcal{H}_0\left(\mathcal{K}_\bullet(*)_{|X}\right)\right) = r$$

With such choices of homogeneous regular sequences $f_1, \ldots, f_r$ and by sheafification of the graded version of Foxby's construction we get the following:

Theorem 1.7. Let X be quasi-projective (*Cohen-Macaulay*) scheme over $Spec(A)$. Let $\mathcal{G}_\bullet$:

$$\mathcal{G}_{k+1} \longrightarrow \mathcal{G}_k \xrightarrow[\partial_k]{} \cdots \longrightarrow \mathcal{G}_r \xrightarrow[\partial_r]{} \mathcal{G}_{r-1} \longrightarrow \cdots \longrightarrow \mathcal{G}_0 \longrightarrow \mathcal{G}_{-1}$$

be a complex of coherent $\mathcal{O}_X$-modules. Assume

$$\forall\, i \in \mathbb{Z},\ \ Y_i := Supp(\mathcal{H}_i(\mathcal{G}_\bullet)), \quad grade(\mathcal{O}_{Y_i}, \mathcal{O}_X) \geq k$$

Then, $\exists \quad \mathcal{L}_\bullet : 0 \longrightarrow \mathcal{L}_k \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow 0 \quad$ a complex of locally free sheaves and a morphism $\nu_\bullet : \mathcal{L}_\bullet \longrightarrow \mathcal{G}_\bullet$, of complexes, such that

1. $\mathcal{H}_0(\nu) : \mathcal{H}_0(\mathcal{L}_\bullet) \twoheadrightarrow \mathcal{H}_0(\mathcal{G}_\bullet)$ is surjective.
2. $\mathcal{L}_\bullet$ is a locally free resolution of $\mathcal{H}_0(\mathcal{L}_\bullet)$.
3. And,

$$grade\,(\mathcal{H}_0(\mathcal{L}_\bullet)) = \dim_{\mathscr{V}(X)}(\mathcal{H}_0(\mathcal{L}_\bullet)) = k \tag{2}$$

1.1 Derived Equivalences

Using Theorem 1.7, we prove some results on Derived equivalences, which has wide applications in K-Theory.

Notations 1.8. Let X will denote a noetherian scheme, with $\dim X = d$.

1. Use the usual notations $Coh(X)$, $\mathscr{V}(X)$ etc.

   For integers $k \geq 0$, denote

$$\begin{cases} Coh^k(X) := \{\mathcal{F} \in Coh(X) : grade\,(\mathcal{F}, \mathcal{O}_X) \geq k\} \\ \mathbb{M}(X) := \{\mathcal{F} \in Coh(X) : \dim_{\mathscr{V}(X)}(\mathcal{F}) < \infty\} \\ \mathbb{M}^k(X) := \{\mathcal{F} \in \mathbb{M}(X) : grade\,(\mathcal{F}, \mathcal{O}_X) \geq k\} \\ C\mathbb{M}^k(X) := \\ \{\mathcal{F} \in \mathbb{M}(X) : grade\,(\mathcal{F}, \mathcal{O}_X) = \dim_{\mathscr{V}(X)}(\mathcal{F}) = k\} \end{cases}$$

   So, we have a filtrations of $\mathbb{M}(X) = \mathbb{M}^0(X)$ and $Coh(X) = Coh^0(X)$, as follows

$$\begin{cases} \mathbb{M}^0(X) \hookleftarrow \mathbb{M}^1(X) \hookleftarrow \cdots \hookleftarrow \mathbb{M}^d(X) \hookleftarrow 0 \\ \\ Coh^0(X) \hookleftarrow Coh^1(X) \hookleftarrow \cdots \hookleftarrow Coh^d(X) \hookleftarrow 0 \end{cases}$$

This would induce filtrations on other associated categories.

2. The categories $Coh(X)$, $\mathbb{M}^k(X)$, $C\mathbb{M}^k(X)$ are exact categories. For exact categories $\mathscr{E}$, the bounded derived category $\mathcal{D}^b(\mathscr{E})$ is obtained as follows:
   (a) The objects of $\mathcal{D}^b(\mathscr{E})$ is same as that of the category $Ch^b(\mathscr{E})$ of bounded chain complexes.
   (b) Morphisms of $\mathcal{D}^b(\mathscr{E})$ are obtained by inverting the quasi-isomorphism in $Ch^b(\mathscr{E})$.
3. For $\mathscr{E} = \mathscr{V}(X), \mathbb{M}(X)$, and integers $k \geq 0$,
$$\left\{ \mathscr{D}^k(\mathscr{E}) := \left\{\mathcal{F}_\bullet \in \mathcal{D}^b(\mathscr{E}) : \forall\ i\ \mathcal{H}_i(\mathcal{F}_\bullet) \in Coh^k(X)\right\}\right.$$
would denote the full subcategory of such objects. (*Note the difference between two fonts* $\mathcal{D}, \mathscr{D}$.)

The Main Theorem on Derived Equivalences:

Theorem 1.9. Let X be a noetherian quasi-projective scheme over an affine scheme Spec(A) and $k \geq 0$ be a fixed integer. Consider the commutative diagram of functors of derived categories:

$$\begin{array}{ccccccc}
\mathcal{D}^b(C\mathbb{M}^{k+1}(X)) & \xrightarrow[\sim]{\zeta} & \mathcal{D}^b(\mathbb{M}^{k+1}(X)) & \xrightarrow[\sim]{\iota} & \mathscr{D}^{k+1}(\mathbb{M}(X)) & \xleftarrow[\sim]{\iota'} & \mathscr{D}^{k+1}(\mathscr{V}(X)) \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \eta \\
\mathcal{D}^b(C\mathbb{M}^{k}(X)) & \xrightarrow[\zeta]{\sim} & \mathcal{D}^b(\mathbb{M}^{k}(X)) & \xrightarrow[\iota]{\sim} & \mathscr{D}^{k}(\mathbb{M}(X)) & \xleftarrow[\iota']{\sim} & \mathscr{D}^{k}(\mathscr{V}(X))
\end{array} \quad (3)$$

Then, all the horizontal functors are equivalences of derived categories and all the vertical functors are fully faithful.

2 Application of (1.9) to Negative $\mathbb{K}$-theory

Theorem 1.9 applies to $\mathbb{K}$-theory (*and Grothendieck Witt theory*).

2.1 Background on K-theory

Suppose $\mathscr{E}$ is an exact category. The Quillen defined the $\mathbf{K}$-theory space through the following steps of constructions:

$$\mathcal{E} \quad \mapsto \quad \mathcal{QE} \quad \mapsto \quad B(\mathcal{QE}) \quad \mapsto \quad \mathbf{K}(\mathcal{E}) := \Omega(B(\mathcal{QE}))$$

where $\mathbf{K}(\mathcal{E})$ is a topological space and defined

$$\forall\ i \geq 0 \quad K_i(\mathcal{E}) := \pi_i(\mathbf{K}(\mathcal{E})) = \pi_{i+1}(B\mathcal{Q}(\mathcal{E})).$$

Theorem 2.1 (Localization)**.** Suppose

$\mathscr{B} \hookrightarrow \mathscr{A} \twoheadrightarrow \mathscr{C}$ is an exact sequence of abelian categories

Then, the induced sequence

$\mathbf{K}(\mathscr{B}) \longrightarrow \mathbf{K}(\mathscr{A}) \longrightarrow \mathbf{K}(\mathscr{C})$ is a homotopy fibration.

Consequently, there is a long exact sequence

$$\cdots \longrightarrow K_n(\mathscr{B}) \hookrightarrow K_n(\mathscr{A}) \twoheadrightarrow K_n(\mathscr{C}) \longrightarrow K_{n-1}(\mathscr{B}) \longrightarrow$$

Remark: Suppose X is a noetherian scheme, $Y \subseteq X$ is a closed sunset and $U = X \setminus Y$.

1. Theorem 2.1, is often applied for the sequence

$$Coh(Y) \longrightarrow Coh(X) \longrightarrow Coh(U)$$

   Consequently, we get a long exact sequence of G-groups

$$\cdots \longrightarrow G_n(Y) \hookrightarrow G_n(X) \twoheadrightarrow G_n(U) \longrightarrow G_{n-1}(Y) \longrightarrow$$

2. A point is ,that it does not apply to the sequence

$$\mathscr{V}(Y) \qquad \mathscr{V}(X) \longrightarrow \mathscr{V}(U)$$

3. This justifies the place on negative $\mathbb{K}$-Theory, also known as non-connective $\mathbb{K}$-Theory.

2.2 Negative $\mathbb{K}$-theory

Our standard reference for K-theory is Schlichting [S2].

1. $\mathbb{K}$-Theory spectra was defined for exact categories $\mathscr{E}$, which is a sequence of topological spaces with bonding maps.

2. $\mathbb{K}$-Theory spectra was also defined for complicial exact categories $\mathscr{E}$ with weak equivalences. We will be thinking of $(Ch^b(\mathscr{E}), \mathscr{Q})$, where $\mathscr{Q}$ denotes the set of all quasi isomorphisms.

   This is also known as negative $\mathbb{K}$-theory or non-connective $\mathbb{K}$-theory.

3. For such a category, $\mathbb{K}(\mathscr{E})$ will denote the $\mathbb{K}$-theory spectra of $\mathscr{E}$ and $\mathbb{K}_i(\mathscr{E})$ will denote the $\mathbb{K}$-groups. Likewise, $\mathbf{K}(\mathscr{E})$ would denote the $\mathbf{K}$-theory space of $\mathscr{E}$ (*as was defined by Quillen*).

   I have a preference to state the non-connective version of $\mathbb{K}$-Theory, and skip the connective version (K-Theory).

Before stating the main application of (1.9) to $\mathbb{K}$-theory, we have the following notations:

$$\begin{cases} \mathscr{C}h^k(\mathscr{V}(X)) := \left\{\mathcal{F}_\bullet \in Ch^b(\mathscr{V}(X)) : \forall\ i\ \mathcal{H}_i(\mathcal{F}_\bullet) \in Coh^k(X)\right\} \\ \mathscr{C}h^k(\mathbb{M}(X)) := \left\{\mathcal{F}_\bullet \in Ch^b(\mathbb{M}(X)) : \forall\ i\ \mathcal{H}_i(\mathcal{F}_\bullet) \in Coh^k(X)\right\} \\ \mathscr{D}^k(\mathscr{V}(X)) := \left\{\mathcal{F}_\bullet \in \mathcal{D}^b(\mathscr{V}(X)) : \forall\ i\ \mathcal{H}_i(\mathcal{F}_\bullet) \in Coh^k(X)\right\} \\ \mathscr{D}^k(\mathbb{M}(X)) := \left\{\mathcal{F}_\bullet \in \mathcal{D}^b(\mathbb{M}(X)) : \forall\ i\ \mathcal{H}_i(\mathcal{F}_\bullet) \in Coh^k(X)\right\} \end{cases}$$

$$\begin{cases} \mathscr{C}h^k(\mathscr{V}(X)) := \left(\mathscr{C}h^k(\mathscr{V}(X)), \mathscr{Q}\right) \\ \mathscr{C}h^k(\mathbb{M}(X)) := \left(\mathscr{C}h^k(\mathbb{M}(X)), \mathscr{Q}\right) \end{cases}$$ likewise, denote

the complicial exact categories with weak equivalences.

Also, For a noetherian scheme X, denote

$$X^{(k)} := \{Y \in X : co\dim(Y) = k\} \quad \text{and} \quad X_x := \mathrm{Spec}(\mathcal{O}_{X,x}).$$

Theorem 2.2. Suppose X is a Cohen-Macaulay quasi-projective scheme over an affine scheme Spec(A) and $k \geq 0$ is an integer. Consider the diagram of $\mathbb{K}$-theory spectra and maps:

$$\begin{array}{ccccc}
\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) & & \mathbb{K}\left(C\mathbb{M}^{k}(X)\right) & & \coprod_{x\in X^{(k)}}\mathbb{K}\left(C\mathbb{M}^{k}\left(X_x\right)\right) \\
\wr\downarrow & & \wr\downarrow\Psi_0 & & \downarrow\wr \\
\mathbb{K}\left(Ch^b\left(C\mathbb{M}^{k+1}(X)\right)\right) & & \mathbb{K}\left(Ch^b\left(C\mathbb{M}^{k}(X)\right)\right) & & \coprod_{x\in X^{(k)}}\mathbb{K}\left(Ch^b\left(C\mathbb{M}^{k}\left(X_x\right)\right)\right) \\
\wr\downarrow & & \wr\downarrow\overline{\zeta} & & \downarrow\wr \\
\mathbb{K}\left(Ch^b\left(\mathbb{M}^{k+1}(X)\right)\right) & \longrightarrow & \mathbb{K}\left(Ch^b\left(\mathbb{M}^{k}(X)\right)\right) & \longrightarrow & \coprod_{x\in X^{(k)}}\mathbb{K}\left(Ch^b\left(\mathbb{M}^{k}\left(X_x\right)\right)\right) \\
\wr\downarrow & & \wr\downarrow\overline{\iota} & & \downarrow\wr \\
\mathbb{K}\left(\mathscr{C}h^{k+1}\left(\mathbb{M}(X)\right)\right) & \longrightarrow & \mathbb{K}\left(\mathscr{C}h^{k}\left(\mathbb{M}(X)\right)\right) & \longrightarrow & \coprod_{x\in X^{(k)}}\mathbb{K}\left(\mathscr{C}h^{k}\left(\mathbb{M}\left(X_x\right)\right)\right) \\
\wr\uparrow & & \wr\uparrow\overline{\iota}' & & \uparrow\wr \\
\mathbb{K}\left(\mathscr{C}h^{k+1}\left(\mathscr{V}(X)\right)\right) & \longrightarrow & \mathbb{K}\left(\mathscr{C}h^{k}\left(\mathscr{V}(X)\right)\right) & \longrightarrow & \coprod_{x\in X^{(k)}}\mathbb{K}\left(\mathscr{C}h^{k}\left(\mathscr{V}\left(X_x\right)\right)\right)
\end{array}$$

Then, the vertical maps are homotopy equivalences of $\mathbb{K}$-theory spectra. Further,the third, fourth, fifth lines are homotopy fibrations of $\mathbb{K}$-theory spectra. In particular, the top line is a zig-zag sequence of homotopy fibration of $\mathbb{K}$-theory spectra, of exact categories:

$$\mathbb{K}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}\left(C\mathbb{M}^{k}(X)\right) \longrightarrow \coprod_{x\in X^{(k)}}\mathbb{K}\left(C\mathbb{M}^{k}\left(X_x\right)\right)$$

Proof. Ψ_0 is an equivalence by Agreement Theorem. Other vertical maps are Homotopy equivalences because of Theorem 1.9. The fifth line is equivalence because for the following well know decomposition Lemma 2.3. Hence so are hence third and fourth. ∎

Lemma 2.3. Suppose X is a Cohen-Macaulay quasi-projective scheme over an affine scheme Spec(A) and $k \geq 0$ is an integer. Then, the sequence of derived categories

$$\mathscr{D}^{k+1}(\mathscr{V}(X)) \longrightarrow \mathscr{D}^{k}(\mathscr{V}(X)) \longrightarrow \coprod_{x \in X^{(k)}} \mathcal{D}^{k}(\mathscr{V}(X_x))$$

is exact up to factor. If X is regular, this sequence is exact.

It is customary to write down the following $\mathbb{K}$-theory exact sequence, which is an immediate consequence of (2.2).

Corollary 2.4. Let X and k be as in Theorem 2.2. Assume X is Cohen-Macaulay. Then, for any integer n, there is an exact sequence of $\mathbb{K}$-groups,

$$\cdots \longrightarrow \mathbb{K}_n\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{K}_n\left(C\mathbb{M}^{k}(X)\right) \longrightarrow \oplus_{x\in X^{(k)}}\mathbb{K}_n\left(\mathbb{M}^{k}(X_x)\right)$$

$$\longrightarrow \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X)\right) \longrightarrow \cdots$$

Proof. Follows from Theorem 2.2. The proof is complete.■

Remark 2.5. Let X be as in Theorem 2.2. Assume X is Cohen-Macaulay. The following are some remarks.

The diagram to compute the Gersten complex, reduces to

$$\begin{array}{ccccc}
\oplus_{x\in X^{(k-1)}}\mathbb{K}_{n+1}\left(C\mathbb{M}^{k-1}(X_x)\right) & & & & \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+2}(X)\right) \\
\downarrow & \searrow & & & \downarrow \\
\mathbb{K}_n\left(C\mathbb{M}^{k}(X)\right) & \longrightarrow & \oplus_{x\in X^{(k)}}\mathbb{K}_n\left(C\mathbb{M}^{k}(X_x)\right) & \longrightarrow & \mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X)\right) \\
\downarrow & & & \searrow & \downarrow \\
\mathbb{K}_n\left(C\mathbb{M}^{k-1}(X)\right) & & & & \oplus_{x\in X^{(k+1)}}\mathbb{K}_{n-1}\left(C\mathbb{M}^{k+1}(X_x)\right)
\end{array}$$

The dotted diagonal arrows form the Gersten complex.

The spectral sequence given in [B3] takes the following form:

$$E_1^{p,q} = \bigoplus_{x \in X^{(p)}} \mathbb{K}_{-p-q}(C\mathbb{M}^p(X_x)) \Longrightarrow \mathbb{K}_{-n}(\mathscr{V}(X)) \qquad \text{along} \quad p+q = n.$$

3 Grothendieck-Witt theory

To do Grothendieck-Witt theory, one needs to incorporate dualities, to what is said above. We refer back to the diagram 3, in Theorem 1.9. The categories

$$\mathbb{M}^k(X),\ Ch^b(\mathbb{M}^k(X)),\ \mathcal{D}^b(\mathbb{M}^k(X)),\ \mathbb{M}(X),\ \mathscr{D}^k(\mathbb{M}(X)),\ \cdots$$

have NO natural duality,

However:

Lemma 3.1. Let X be a noetherian scheme and $k \geq 0$.

1. Then, $\mathscr{C}h^b(\mathscr{V}(X))$ and $\mathscr{D}^k(\mathscr{V}(X))$ have a duality induced by $\mathcal{H}om\,(-, \mathcal{O}_X)$.

2. $\mathcal{F} \mapsto \mathcal{E}xt^k\,(\mathcal{F}, \mathcal{O}_X)$
   is a duality on $C\mathbb{M}^k(X)$.
   It induces a duality on $Ch^b(C\mathbb{M}^k(X))$ and $\mathcal{D}^b\left(C\mathbb{M}^k(X)\right)$.

3. There is no natural functor form $Ch^b(C\mathbb{M}^k(X))$ to $\mathscr{C}h^b(\mathscr{V}(X))$.
   The functor $\mathcal{D}^b(C\mathbb{M}^k(X)) \xrightarrow{\sim} T^k\mathscr{D}^k(\mathscr{V}(X))$ is a duality preserving equivalence.

This makes us go through the category $\mathscr{P}erf(X)$ of perfect complexes, which has a duality $\mathcal{F}_\bullet \mapsto Hom\,(\mathcal{F}_\bullet, I_\bullet)$ where $I_\bullet$ is an injective resolution of $\mathcal{O}_X$.

The following is a diagram of equivalences, analogous to the diagram in Theorem 1.9.

Corollary 3.2. Suppose X is a quasi-projective scheme over an affine scheme Spec(A), and $k \geq 0, r$ are integers. Consider the diagram

$$dgC\mathbb{M}^k(X) \longrightarrow T^k dg\mathcal{P}erf^k(X) \longleftarrow T^k dg^k \mathscr{V}(X) \tag{4}$$

Intuitivey, think of:

$$Ch^b\left(C\mathbb{M}^k(X)\right) \longrightarrow T^k\mathcal{P}erf^k(X) \longleftarrow T^k\mathscr{C}h^k\left(\mathscr{V}(X)\right) \tag{5}$$

In this diagram, all the arrow are functors preserving quasi isomorphisms and duality. Further, the horizontal arrows induce equivalences of the associated triangulated categories:

$$\mathcal{D}^b\left(C\mathbb{M}^k(X)\right) \longrightarrow T^k\mathcal{D}^b\left(\mathcal{P}erf^k(X)\right) \longleftarrow T^k\mathscr{D}^k\left(\mathscr{V}(X)\right) \tag{6}$$

Main Theorem in GW-Theorey:

Theorem 3.3. *Suppose X be Cohen-Macaulay quasi projective scheme, k, r are as above. Assume further that X is a Cohen-Macaulay scheme. Consider the following diagram of $\mathbb{G}W$-Bispetra:*

$$\begin{array}{ccccc}
\mathbb{G}W^{[-1+r]}\left(dgC\mathbb{M}^{k+1}(X)\right) & & \mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}(X)\right) & & \coprod_{x\in X^{(k)}}\mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}\left(X_x\right)\right) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{G}W^{[k+r]}\left(dgPerf^{k+1}(X)\right) & & \mathbb{G}W^{[k]}\left(dgPerf^{k}X)\right) & & \coprod_{x\in X^{(k)}}\mathbb{G}W^{[k+r]}\left(dgPerf(X_x)\right) \\
\uparrow & & \uparrow & & \uparrow \\
\mathbb{G}W^{[k+r]}\left(dg^{k+1}\mathscr{V}(X)\right) & \longrightarrow & \mathbb{G}W^{[k+r]}\left(dg^{k}\mathscr{V}(X)\right) & \longrightarrow & \coprod_{x\in X^{(k)}}\mathbb{G}W^{[k+r]}\left(dg^{k}\mathscr{V}(X_x)\right)
\end{array}$$

In this diagram, all the vertical arrows are equivalence of homotopy Bispectra and the bottom sequence is a homotopy fibration of bispectra. In particular, there is a sequence zig-zag maps of Bispectra

$$\mathbb{G}W^{[-1+r]}\left(dgC\mathbb{M}^{k+1}(X)\right) \longrightarrow \mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}(X)\right) \longrightarrow \coprod_{x\in X^{(k)}}\mathbb{G}W^{[r]}\left(dgC\mathbb{M}^{k}\left(X_x\right)\right)$$

that is a homotopy fibration.

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