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Obstruction theory in algebra and topology

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Abstract. Suppose $X = Spec(A)$ is a real smooth affine variety of dimension $n \geq 2$ and M is the manifold of real points of X. Assume X is orientable and M is nonempty. In this paper, we prove that there is a natural homomorphism $\zeta : E(A, A) \to H^n(M, \mathbb{Z})$ from the Euler class group to the singular cohomology group.

1. Introduction

The following is a classical theorem in obstruction theory in topology.

Theorem 1.1 (see theorem 12.5 in [9]). *Let M be an orientable real manifold of dimension n and $\mathcal{E}$ be an orientable real vector bundle of rank n on M. Then $\mathcal{E}$ has a nowhere vanishing section if and only if the Euler class $e(\mathcal{E}) = 0$.*

There is a lot of similarities between the study of vector bundles over real manifolds M and the study of finitely generated projective modules over noetherian commutative rings A. This is because there is an equivalence (see [15,16]) between the category of vector bundles over a finite dimensional connected paracompact space M (resp. category of algebraic vector bundles over $Spec(A)$) and the category of finitely generated projective modules over the ring $C(M)$ of real valued continuous functions on M (resp. over A). The correspondance is given by $\mathcal{E} \to P(\mathcal{E})$ where $\mathcal{E}$ is a vector bundle over M (resp. an algebraic vector bundle over $Spec(A)$) and $P(\mathcal{E})$ is the module of sections of $\mathcal{E}$.

More often than not, progress in topology has guided a lot of research in projective modules. The obstruction theory in topology is well developed,

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beautiful and is classical. Advent of obstruction theory in algebra is a more recent development. After the work of Mohan Kumar and Murthy ([11,10,12]) on obstruction theory for affine varieties over algebraically closed fields, Nori (see [7]) outlined a program for obstruction theory for smooth affine varieties $Spec(A)$ over infinite fields. This program of Nori flourished beyond all expectations at the time when it was introduced in early nineties ([7]).

While the program of Nori flourished, attempts have often been to mimick the results in topology. This is unavoidable when we work on abstract affine algebraic varieties. For real smooth algebraic varieties, it begs the question whether we can reconcile the obstruction theory in algebra and topology. In particular, whether there is a natural homomorphism from the obstruction groups in algebra to the obstruction groups in topology. In this paper, *we address this issue for oriented smooth real affine varieties $X = Spec(A)$ and oriented vector bundles V over X with $rank(V) = \dim X = n \geq 2$.*

We refer to ([2]) for basic definitions and other facts on obstruction theory in algebra. The obstruction groups in algebra are commonly known as Euler class groups. Given a noetherian commutative ring A with $\dim A = n \geq 2$, and a projective A-module L of rank one, there is a Euler class group $E(A, L)$ defined. Also, for a projective $A-$module P of $rank(P) = n$ with determinant L and for an isomorphism (orientation) $\chi : L \xrightarrow{\sim} \wedge^n P$ there is an Euler class $e(P, \chi) \in E(A, L)$ defined.

Let $X = Spec(A)$ be an oriented real smooth affine variety with $\dim X = n \geq 2$ and M be the manifold of real points of X. Assume M is nonempty. Also let $\mathbb{R}(X) = S^{-1}A$, where S is the multiplicative set of all $f \in A$ that do not vanish at any real point of X. In this paper, we define a natural isomorphism

$$\zeta : E(\mathbb{R}(X), \mathbb{R}(X)) \xrightarrow{\sim} H^n(M, \mathbb{Z})$$

where $H^n(M, \mathbb{Z})$ is the singular cohomology group of M with $\mathbb{Z}$ coefficients. Now suppose P is a projective $\mathbb{R}(X)-$module of rank n with trivial determinant and $\chi : \mathbb{R}(X) \xrightarrow{\sim} \wedge^n P$ is an orientation. Let $V = V(P)$ be the vector bundle over M whose module of sections is given by $P \otimes C(M)$. Then, we prove that Euler classes agree, i.e. $\zeta(e(P, \chi)) = e(V^*, \chi')$ where e denotes the Euler class (algebraic or topological) and χ' is the orientation induced by χ.

In a subsequent paper ([8]), we address the case of nonorientable situation and reconcile the obstruction theories in algebra and topology for the general case, by defining similar natural isomorphsms between obstruction groups.

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2. Preliminaries

In this section, we recall some definitions and notations and also give some preliminaries from topology. For the benefit of the readers, we recall the following version (see [2]) of the definition of the Euler class groups.

Definition 2.1. *Let $A = S^{-1}B$ be a localization of smooth affine algebra B over a field k, with $\dim A = \dim B = n \geq 2$. We assume that $char(k) = 0$ and all maximal ideals of A has height n. Let L be a projective module of rank one.*

1. *Let G be the free abelian group generated by the set of all pairs (m, ω), where m is a maximal ideal of A and $\omega : L/mL \xrightarrow{\sim} \wedge^n m/m^2$ is an isomorphism. Let $I = m_1 \cap \cdots \cap m_r$ be an intersection of finitely many maximum ideals and $\omega_I : L/IL \xrightarrow{\sim} \wedge^n I/I^2$ be an isomorphism. Such an ω_I is called a local orientation on I. Let $(I, \omega_I) := \sum_{i=1}^{r}(m_i, \omega_i) \in G$, where ω_i is the local orientation on m_i induced by ω_I. Such an element (I, ω_I) is called global, if it is induced by a surjective homomorphism $f : L \oplus A^{n-1} \twoheadrightarrow I$. Let H be the subgroup of G generated by global elements (I, ω_I) of G. Define Euler class group of A relative to L as $E(A, L) := G/H$. (The image of (I, ω) in $E(A, L)$ will also be denoted by the same notation.)*
2. *Now let P be a projective $A-$module of rank n and $\det P = L$. Let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism (orientation). Suppose $f : P \twoheadrightarrow I$ is a surjective homomorphism, where $I = m_1 \cap \cdots \cap m_r$ is intersection of finitely many maximum ideals. Let $\varphi : \wedge^n(P/IP) \xrightarrow{\sim} \wedge^n(I/I^2)$ be the isomorphism induced by f. Define the Euler class $e(P, \chi) = (I, \varphi\overline{\chi}) \in E(A, L)$.*
3. *We comment that in our case of localizations of smooth affine rings A as above, these definitions are equivalent to those in ([2]). This is because, Swan's Bertini theorem remains valid for such rings (see [2, Remark 4.7]).*
4. *We also comment that the conjecture of Nori that P has an unimodular element if and only if $e(P, \chi) = 0$, was proved by Bhatwadekar and Sridharan ([2]).*

We include the following standard notations that will be useful for our subsequent discussions.

Notation 2.2.

1. For a noetherian commutative ring A and $X = Spec(A)$, the Grothendieck group of finitely generated projective $A-$modules will be denoted by $K_0(A)$ or $K_0(X)$ and the Chow group of zero cycles of X modulo rational equivalence will be denoted by $CH_0(A)$ or $CH_0(X)$.

2. Suppose X is a smooth real manifold with $\dim X = n$. Then $H_r(X, R)$, $H^r(X, R)$ would, respectively, denote the singular homology and cohomology group with $R-$coefficients. Assume X is oriented and let (V, χ) be an oriented vector bundle over X of rank r. Then, $e(V, \chi) \in H^r(X, \mathbb{Z})$ will denote the Euler class of (V, χ) (see [9]). Note that we use the same notation e for Euler classes in both algebraic context and topological context. Also, $KO(X)$ will denote the Grothendieck group of real vector bundles over X.

2.1 Local index

In this subsection we would recall the definition of local index from topology. First, we recall the definition of degree of the continuous maps of spheres.

Definition 2.3. *Suppose $f : \mathbb{S}^n \to \mathbb{S}^n$, $n \geq 1$ is a continuous function. Then f induces an endomorphism*

$$H_n(f) : H_n(\mathbb{S}^n, \mathbb{Z}) \cong \mathbb{Z} \to H_n(\mathbb{S}^n, \mathbb{Z}) \cong \mathbb{Z}$$

of the homology group $H_n(\mathbb{S}^n, \mathbb{Z})$. The degree of f is defined as

$$\deg(f) = H_n(f)(1) \in \mathbb{Z}.$$

Clearly $\deg(\mathrm{id}_{\mathbb{S}^n}) = 1$. *From ([5, 16.1]), we also have* $\deg(\rho|_{\mathbb{S}^n}) = -1$, *for the reflection map* $\rho(x_0, x_1, x_2, \ldots, x_n) := (-x_0, x_1, x_2, \ldots, x_n)$.

The following is a lemma from elementary topology.

Lemma 2.4. *Suppose $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a linear isomorphism and*

$$\left.\frac{\varphi}{||\varphi||}\right|_{\mathbb{S}^n} : \mathbb{S}^n \to \mathbb{S}^n$$

is the restriction of the normalized φ, then we have

$$\deg\left(\left.\frac{\varphi}{||\varphi||}\right|_{\mathbb{S}^n}\right) = \frac{\det(\varphi)}{|\det(\varphi)|} = \pm 1$$

the sign of $\det(\varphi)$.

Proof. Indeed, recall that the group manifold $GL_{n+1}(\mathbb{R})$ is the union of two path-connected components $\det^{-1}(0, \infty)$ and $\det^{-1}(-\infty, 0)$ and hence $\varphi_0, \varphi_1 \in GL_{n+1}(\mathbb{R})$ are connected by a (continuous) path in $GL_{n+1}(\mathbb{R})$ if and only if $\det(\varphi_0)$ and $\det(\varphi_1)$ are of the same $\pm$-sign. Note that if

$\varphi_0, \varphi_1 \in GL_{n+1}(\mathbb{R})$ are connected by a path $\varphi_t \in GL_{n+1}(\mathbb{R})$ continuous in $t \in [0,1]$, then

$$t \mapsto f_t := \left.\frac{\varphi_t}{\|\varphi_t\|}\right|_{\mathbb{S}^n}$$

is a homotopy from $\frac{\varphi_0}{\|\varphi_0\|}\big|_{\mathbb{S}^n}$ to $\frac{\varphi_1}{\|\varphi_1\|}\big|_{\mathbb{S}^n}$ via continuous functions $f_t : \mathbb{S}^n \to \mathbb{S}^n$, and hence $H_n\big(\frac{\varphi_0}{\|\varphi_0\|}\big|_{\mathbb{S}^n}\big) = H_n\big(\frac{\varphi_1}{\|\varphi_1\|}\big|_{\mathbb{S}^n}\big)$, which implies $\deg\big(\frac{\varphi_0}{\|\varphi_0\|}\big|_{\mathbb{S}^n}\big) = \deg\big(\frac{\varphi_1}{\|\varphi_1\|}\big|_{\mathbb{S}^n}\big)$. For $\varphi \in GL_{n+1}(\mathbb{R})$ with $\det(\varphi) > 0$, we have φ and id in the same path connected component of $GL_{n+1}(\mathbb{R})$, and hence

$$\deg\left(\left.\frac{\varphi}{\|\varphi\|}\right|_{\mathbb{S}^n}\right) = \deg\left(\left.\frac{\mathrm{id}}{\|\mathrm{id}\|}\right|_{\mathbb{S}^n}\right) = \deg(\mathrm{id}_{\mathbb{S}^n}) = 1.$$

For $\varphi \in GL_{n+1}(\mathbb{R})$ with $\det(\varphi) < 0$, we have φ and ρ in the same path connected component of $GL_{n+1}(\mathbb{R})$ and hence

$$\deg\left(\left.\frac{\varphi}{\|\varphi\|}\right|_{\mathbb{S}^n}\right) = \deg\left(\left.\frac{\rho}{\|\rho\|}\right|_{\mathbb{S}^n}\right) = \deg(\rho|_{\mathbb{S}^n}) = -1.$$

Thus we have $\deg\big(\frac{\varphi}{\|\varphi\|}\big|_{\mathbb{S}^n}\big)$ equal to the sign of $\det(\varphi)$ for any $\varphi \in GL_{n+1}(\mathbb{R})$. The proof is complete. □

From ([6, Section 8.9]), we may take the following definition of local index, which turns out to be independent of the choice of local parametrizations used.

Definition 2.5. *Let B be a smooth oriented manifold of dimension n. Let $v \in B$ be a point and V be an open neighborhood of v. Suppose*

$$f = (f_1, \ldots, f_n) : V \to \mathbb{R}^n$$

is an ordered n-tuple of smooth functions such that f has an isolated zero at v. Now fix a parametrization $\varphi : \mathbb{R}^n \xrightarrow{\sim} U$, compatible with the orientation of B, where $U \subseteq V$ is a neighbourhood of $v = \varphi(0)$. By modifying φ, we assume that $f\varphi$ vanishes only at the origin $0 \in \mathbb{R}^n$. Define index $j_v(f_1, \ldots, f_n)$ to be the degree of the map

$$\frac{\eta}{\|\,\eta\,\|} : \mathbb{S}^{n-1} \to \mathbb{S}^{n-1} \qquad \textit{where} \quad \eta = (f\varphi)|_{\mathbb{S}^{n-1}}.$$

In this paper, we utilize the index of local diffeomorphisms, and the next lemma summarizes what we need.

Lemma 2.6. *Let $v \in B$, $f : V \to \mathbb{R}^n, \varphi : \mathbb{R}^n \xrightarrow{\sim} U$ be as in (2.5) with $\dim B = n \geq 2$. Further assume f is a diffeomorphism onto an open set $f(V) \subset \mathbb{R}^n$. Then*

$$j_v(f) = \deg\left(\left.\frac{D(f\varphi)(0)}{\|D(f\varphi)(0)\|}\right|_{\mathbb{S}^{n-1}}\right) = \frac{\det(D(f\varphi)(0))}{|\det(D(f\varphi)(0))|} = \pm 1$$

where $D(f\varphi)(0)$ *denotes the total derivative of the composite function* $f\varphi$ *at* 0.

Proof. The last two equalities follow directly from lemma 2.4 and so we prove the first equality only.

To simplify the notations used, we replace f by $f\varphi$ and assume $V = \mathbb{R}^n$ with $v = 0$. Then it remains to show that

$$\deg\left(\frac{f}{\|f\|}\bigg|_{\mathbb{S}^{n-1}}\right) = \deg\left(\frac{Df(0)}{\|Df(0)\|}\bigg|_{\mathbb{S}^{n-1}}\right).$$

Note that $Df(0) \in GL_n(\mathbb{R})$ and hence

$$\|Df(0)x\| \geq \delta\|x\|$$

for all $x \in \mathbb{R}^n$ for some constant $\delta > 0$. Since f has the Taylor expansion

$$f(x) = Df(0)x + O(\|x\|^2)$$

at 0, there is $\varepsilon > 0$ such that for all $x \in \varepsilon\mathbb{S}^{n-1}$,

$$\|f(x) - Df(0)x\| = O(\|x\|^2) < \delta\|x\| \leq \|Df(0)x\|$$

and hence

$$f_t(x) := Df(0)x + t(f(x) - Df(0)x) \neq 0$$

for all $t \in [0,1]$. So $f|_{\varepsilon\mathbb{S}^{n-1}} = f_1|_{\varepsilon\mathbb{S}^{n-1}}$ is homotopic to $Df(0)|_{\varepsilon\mathbb{S}^{n-1}} = f_0|_{\varepsilon\mathbb{S}^{n-1}}$ via continuous maps $f_t|_{\varepsilon\mathbb{S}^{n-1}} : \varepsilon\mathbb{S}^{n-1} \to \mathbb{R}^n\backslash\{0\}$. On the other hand with $1, \varepsilon \in (0,\infty)$, it is clear that $f|_{\mathbb{S}^{n-1}}$ is homotopic to $f\psi|_{\mathbb{S}^{n-1}} = f|_{\varepsilon\mathbb{S}^{n-1}}\psi$ as continuous functions from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n\backslash\{0\}$, where $\psi : x \in \mathbb{S}^{n-1} \mapsto \varepsilon x \in \varepsilon\mathbb{S}^{n-1}$.

So we get

$$\deg\left(\frac{f}{\|f\|}\bigg|_{\mathbb{S}^{n-1}}\right) = \deg\left(\frac{f\psi}{\|f\psi\|}\bigg|_{\mathbb{S}^{n-1}}\right) = \deg\left(\frac{Df(0)\psi}{\|Df(0)\psi\|}\bigg|_{\mathbb{S}^{n-1}}\right)$$
$$= \deg\left(\frac{\varepsilon Df(0)}{\|\varepsilon Df(0)\|}\bigg|_{\mathbb{S}^{n-1}}\right) = \deg\left(\frac{Df(0)}{\|Df(0)\|}\bigg|_{\mathbb{S}^{n-1}}\right).$$

The proof is complete. □

We state the following theorem that computes the Euler class of a vector bundle in terms of indices of sections of the vector bundle (see [6, §9.9, Theorem 3]).

Theorem 2.7. *Let* B *be a smooth compact connected oriented manifold of dimension* $n \geq 2$*and* $\xi : E \to B$ *be an oriented vector bundle on* B *of rank*

n. Fix an orientation $\chi : B \times \mathbb{R} \xrightarrow{\sim} \wedge^n E$. Let $\sigma : B \to E$ be a smooth cross section with isolated zeros $v_1, \dots, v_k \in B$ and be transversal to the zero section of E.

For each v_i, we fix a local trivialization $\psi_i : E|_{U_i} \xrightarrow{\sim} U_i \times \mathbb{R}^n$ of E on an open neighborhood U_i of v_i such that ψ_i is compatible with the orientation χ of E and v_i is the only zero of σ in U_i. Defining $\sigma_i := (\psi_i)_2 \sigma|_{U_i}$, where $(\psi_i)_2 : E|_{U_i} \to \mathbb{R}^n$ is the second component function of ψ, we get the Euler class

$$e(E, \chi) = j(\sigma) := \sum_{i=1}^{k} j_{v_i}(\sigma_i) \in \mathbb{Z} \cong H^n(B, \mathbb{Z}).$$

3. Main results

Now we state our main theorem that defines a natural homomorphism from algebraic to topological obstruction groups, in the oriented case.

Theorem 3.1. *Suppose $X = Spec(A)$ is a smooth affine variety over the reals $\mathbb{R}$ with $\dim A = n \geq 2$. Let $X(\mathbb{R})$ be the set of all real maximal ideals of A and M be the smooth manifold of real points in X. Write $\mathbb{R}(X) = S^{-1}A$, where S is the multiplicative set of $f \in A$ that do not vanish on $X(\mathbb{R})$. We assume $X(\mathbb{R}) \neq \phi$ and X is oriented (i.e. $\wedge^n \Omega_{A/\mathbb{R}} \approx A$).*

1. *Let $m \in Spec(\mathbb{R}(X))$ be a maximal ideal and $v \in M$ be the real point corresponding to m. Given a local orientation $\omega : \mathbb{R}(X)/m \xrightarrow{\sim} \wedge^n m/m^2$, we can represent $\omega = \overline{f_1} \wedge \cdots \wedge \overline{f_n}$, where $f_1, \dots, f_n \in m$ generate m/m^2. Using (2.5), define $j(m, \omega) = j_v(f_1, \dots, f_n)$. Then, the definition of $j(m, \omega)$ is independent of $f_1, \dots, f_n$. Further $j(m, \omega) = \pm 1$.*
2. *Let $C_1, \dots, C_r$ be the compact connected components of M. The association $(m, \omega) \to j(m, \omega) \in H^n(C_i, \mathbb{Z}) = \mathbb{Z}$, for m corresponding to points in C_i, and $(m, \omega) \to 0$ otherwise, induces a homomorphism*

$$\zeta : E(\mathbb{R}(X), \mathbb{R}(X)) \to H^n(M, \mathbb{Z}).$$

3. *In fact, ζ is an isomorphism.*
4. *Suppose P is a projective $\mathbb{R}(X)$-module of rank n with trivial determinant and $\chi : \mathbb{R}(X) \xrightarrow{\sim} \wedge^n P$ is an orientation. Let $V(P)$ be the corresponding real vector bundle on M, whose module of sections is given by $P \otimes C(M)$. Then $\zeta(e(P, \chi)) = e(V(P)^*, \chi')$, where χ' denotes the orientation on $V(P)^*$ induced by χ.*

Proof. Let (m, ω) be a local orientation and $v \in M$ be the point of m. Let $\omega = \overline{f_1} \wedge \cdots \wedge \overline{f_n} = \overline{g_1} \wedge \cdots \wedge \overline{g_n}$, where $f_i, g_i \in m$ are two sets of generators of m/m^2.

So, in a neighborhood $D(s)$, where $s \in 1 + m^2$, we have

$$\begin{pmatrix} f_1 \\ f_2 \\ \cdots \\ f_n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} g_1 \\ g_2 \\ \cdots \\ g_n \end{pmatrix} \qquad where \quad a_{ij} \in \mathbb{R}(X)_s.$$

We write $\alpha = (a_{ij})$. We have, the diagram

$$\begin{tikzcd} R_s^n \arrow[r, two heads, "(f)"] \arrow[d, "\alpha"'] & m_s \\ R_s^n \arrow[ur, two heads, "(g)"'] & \end{tikzcd} \quad reduces\ to \quad \begin{tikzcd} (R/m)^n \arrow[r, "(f)", "\sim"'] \arrow[d, "\overline{\alpha}"'] & m/m^2 \\ (R/m)^n \arrow[ur, "\sim", "(g)"'] & \end{tikzcd} .$$

Taking $n^{th}-$exterior $\wedge^n$ we have $\det(\alpha) = 1 + H$ where $H \in mR_s$.

Now take neighborhood U of v and a smooth paramatrization $\phi : \mathbb{R}^n \xrightarrow{\sim} U$ with $\phi(0) = v$ that is compatible with the orientation of M. Let us compute the Jacobian matrices $J(f\phi), J(g\phi)$. We denote the coordinate functions of $\mathbb{R}^n$ by $x_1, x_2, \ldots, x_n$ and we have

$$\frac{\partial(f_i\phi)}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial((a_{ik}\phi)(g_k\phi))}{\partial x_j} = \sum_{k=1}^{n} (a_{ik}\phi)\frac{\partial(g_k\phi)}{\partial x_j} + \sum_{k=1}^{n} \frac{\partial(a_{ik}\phi)}{\partial x_j}(g_k\phi).$$

So,

$$J(f\phi) = \beta J(g\phi) + B \quad where \quad \beta = (a_{ik}\phi) \quad and$$

$$B = \left(\sum_{k=1}^{n} \frac{\partial(a_{ik}\phi)}{\partial x_j}(g_k\phi) \right).$$

Evaluating at 0, we have $J(f\phi)(0) = \beta(0)(J(g\phi)(0))$ and $\det \beta(0) = \det \alpha(v) = 1$. Now we prove that $j(m, \omega)$ is independent of (f) or (g) :

1. Write $\varphi = J(f\phi), \gamma = J(g\phi)$. Then

$$(f\phi(x)) = \varphi(0)x + O(\| x \|^2), \quad (g\phi(x)) = \gamma(0)x + O(\| x \|^2)$$

Also

$$\beta(x) = \beta(0) + O(\| x \|).$$

2. Combining, we have

$$f\phi(x) = \beta(0)\gamma(0)x + O(\| x \|^2).$$

3. Since $\det \beta(0) = 1$, by Lemma 2.6, we have

$$j_v(f) = \frac{\det \varphi(0)}{|\det \varphi(0)|} = \frac{\det \gamma(0)}{|\det \gamma(0)|} = j_v(g).$$

So, $j(m, \omega)$ is independent of $f_1, \ldots, f_n$ and $j(m, \omega) = \frac{\det \varphi(0)}{|\det \varphi(0)|} = \pm 1$. This establishes (1) of the theorem.

Now we prove (2). Let $f : \mathbb{R}(X)^n \twoheadrightarrow J$ be a surjective map, where $J = m_1 \cap \cdots \cap m_k$ is intersection of distinct maximal ideals of $\mathbb{R}(X)$. Let ω_i denote the local orientation on m_i, induced by f. We need to prove

$$\eta = \sum_{i=1}^{k} j(m_i, \omega_i) = 0.$$

But f is a section of the trivial bundle T over M of rank n. Let $\chi : M \times \mathbb{R} \xrightarrow{\sim} \wedge^n T$ be the obvious orientation. By applying (2.7), to connected components of M we see that $\eta = e(T, \chi) = 0$ (see [9]). So, (2) is established.

It follows from the structure theorem ([1] or see [3, Theorem 4.21]) that $E(\mathbb{R}(X), \mathbb{R}(X)) = \mathbb{Z}^r$. It is well konwn that for the compact components C_i, we have $H^n(C_i, \mathbb{Z}) = \mathbb{Z}$. For noncompact components C of M, also is well known (see [14, Theorem 2.2]) that $H^n(C, \mathbb{Z}) = 0$. Therefore, $H^n(M, \mathbb{Z}) = \mathbb{Z}^r$. So, (3) follows because $j(m, \omega) = \pm 1$, by (1).

We have $E(\mathbb{R}(X), \mathbb{R}(X)) = \mathbb{Z}^r = \oplus_{i=1}^r \mathbb{Z}\epsilon_i$, where ϵ_i is the generator of $\mathbb{Z}^r$ corresponding to the compact connected components C_i. Suppose $e(P, \chi) = (n_1, \ldots, n_r) \in \mathbb{Z}^r$. Let $p_i = \mid n_i \mid$. Pick distinct points $v_{i1}, \ldots, v_{ip_i} \in C_i$ and let $m_{ik} \subseteq \mathbb{R}(X)$ be the maximal ideal of v_{ik}. (*Note, if $p_i = 0$, the subsequent argument works vacuously. Alternately, we could work with two distinct points with opposite orientations.*) We can attach local orientations ω_{ik} on m_{ik} such that $(m_{ik}, \omega_{ik}) = \pm\epsilon_i$ in $E(\mathbb{R}(X), \mathbb{R}(X))$, according as $n_i > 0$ or $n_i < 0$. Therefore, $e(P, \chi) = (I, \omega) = \sum (m_{ik}, \omega_{ik})$ where $I = \cap_{ik} m_{ik}$ and ω is obtained from ω_{ik}. By ([2, Cor. 4.3]), there is a surjective map $\sigma : P \twoheadrightarrow I$ such that the diagram

$$\begin{tikzcd}
P \arrow[r, two heads, "\sigma"] \arrow[d] & I \arrow[d] \\
P/IP \arrow[r] & I/I^2 \\
F/IF \arrow[u, "\psi"] \arrow[ur, "\eta"'] &
\end{tikzcd}$$

commutes for some ψ, where $F = \mathbb{R}(X)^n$, $\det\psi = \overline{\chi}$. and $\det\eta = \omega$. We have,

$$\zeta(e(P,\chi)) = \sum \zeta(m_{ik},\omega_{ik}) = \sum_i \left(\sum_k j(m_{ik},\omega_{ik})\epsilon_i\right).$$

Now σ is a section of $V(P)^*$ transversal to the zero section and by (2.7),

$$e(V(P)^*_{|C_i},\chi_i) = \sum_j j(m_{ik},\omega_{ik})\epsilon_i,$$

where χ_i is the orientation of $V(P)^*_{|C_i}$ induced by χ'. Therefore, $\zeta(E(P,\chi)) = e(V(P)^*,\chi')$. This complete the proof of (4) and the theorem. $\square$

Corollary 3.2. *With notations as in (3.1), there is an isomorphism* $\zeta_0 : CH_0(\mathbb{R}(X)) \xrightarrow{\sim} H^n(M,\mathbb{Z}/(2))$, *such that the diagram*

$$\begin{tikzcd}
E(\mathbb{R}(X),\mathbb{R}(X)) \arrow[r, "\zeta"] \arrow[d, "\epsilon"] & H^n(M,\mathbb{Z}) \arrow[d, "\epsilon'"] \\
CH_0(\mathbb{R}(X)) \arrow[r, "\zeta_0"] & H^n(M,\mathbb{Z}/(2)).
\end{tikzcd}$$

commutes, where ϵ,ϵ' *are the natural homomorphism.*

Proof. As in the proof of (3.1), let $C_1,\ldots,C_r$ be the connected compact components of M. We have $E(\mathbb{R}(X),\mathbb{R}(X)) = \mathbb{Z}^r$. Also by structure theorem for Chow groups, we have $CH_0(\mathbb{R}(X)) = (\mathbb{Z}/(2))^r$ (use [4] or [1, 4.10]). As in the proof of theorem 3.1, we have $H^n(M,\mathbb{Z}) = \mathbb{Z}^r$ and $H^n(M,\mathbb{Z}/(2)) = (\mathbb{Z}/(2))^r$. The composition map

$$\epsilon'\zeta : E(\mathbb{R}(X),\mathbb{R}(X)) \to H^n(M,\mathbb{Z}/(2)))$$

is surjective and $kernel(\epsilon'\zeta) = 2\mathbb{Z}^r$. Therefore, $\epsilon'\zeta$ factors through an isomorphism $\zeta_0 : CH_0(\mathbb{R}(X)) \xrightarrow{\sim} H^n(X(\mathbb{R}),\mathbb{Z}/(2))$. This completes the proof. $\square$

Corollary 3.3. *We continue to use the notations as in (3.1). Assume that the Picard group* $Pic(A) = 0$. *Then, the following diagram*

$$\begin{tikzcd}
K_0(A) \arrow[r] \arrow[d, "C_0"'] & K_0(\mathbb{R}(X)) \arrow[r] \arrow[d, "C_0"'] & KO(M) \arrow[d, "w_n"] \\
CH_0(A) \arrow[r] & CH_0(\mathbb{R}(X)) \arrow[r, "\zeta_0"] & H^n(M,\mathbb{Z}/(2))
\end{tikzcd}$$

commutes, where C_0 *denotes the top (i.e* n^{th}*) Chern class homomorphism and* w_n *denotes the top Stiefel-Whitney class homomorphism.*

Proof. The first rectangle in the diagram obviously commutes. Also, any element $\tau \in \tilde{K}_0(\mathbb{R}(X))$ can be written as $\tau = [P] - [\mathbb{R}(X)^n]$, where P is a projective $\mathbb{R}(X)-$module of rank n. Since $Pic(A) = 0$, we can fix an orientation $\chi : \mathbb{R}(X) \xrightarrow{\sim} \wedge^n P$. Let $V = V(P)$ be the vector bundle on M, whose module of sections is given by $P \otimes C(M)$ and χ' be the orientation of V induced by χ. By (3.1), $\zeta(e(P^*, (\chi^*)^{-1})) = e(V, \chi')$. By (3.2), we have the commutative diagram

$$\begin{array}{ccc} E(\mathbb{R}(X), \mathbb{R}(X)) & \xrightarrow{\zeta} & H^n(M, \mathbb{Z}) \\ \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle \epsilon'} \\ CH_0(\mathbb{R}(X)) & \xrightarrow{\zeta_0} & H^n(M, \mathbb{Z}/(2)). \end{array}$$

We use the usual convention and note that the top Chern class of P is given by a generic section $f : P^* \twoheadrightarrow J$ (see [12, Remark 3.6]). Now (See [9, Property 9.5]),

$$\begin{aligned} \zeta_0 C_0(\tau) &= \zeta_0(C_0(P)) = \zeta_0(\epsilon(e(P^*, (\chi^*)^{-1}))) \\ &= \epsilon'(\zeta(e(P^*, (\chi^*)^{-1}))) = \epsilon' e(V, \chi') = w_n(V). \end{aligned}$$

The proof is complete. □

Remark 3.4. In Corollary 3.3, we can drop the condition $Pic(A) = 0$, by replacing $K_0(A)$ by $F^2K_0(A)$ and $K_0(\mathbb{R}(X))$ by $F^2K_0(\mathbb{R}(X))$, where for a connected commutative ring R, $F^2K_0(R)$ denotes the subgroup of $\tau \in K_0(R)$ with $rank(\tau) = 0$ and $det(\tau) = 1$.

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