

An overview of Euler class theory

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1 Abstract

The Euler class program was outlined, by M. V. Nori around 1990, as a possible obstruction theory for projective modules of top rank to split off a free direct summand. The program has its genesis in Topology and has been the most important development in the subject of projective modules in its recent history. We give an overview of this theory with an emphasis on the following question:

Question: Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over \mathbb{R} (the field of real numbers) and P a projective A -module of rank n . Under what further restrictions, does vanishing of the top Chern class

$$C^n(P) = 0 \implies P \simeq A \oplus Q?$$

We answer this question completely. We show that, in some cases, additional topological obstruction does exist.

This is a joint work with S. M. Bhatwadekar and Mrinal Kanti Das ([BDM], Invent. Math., to appear). The talk is adapted from my talk in Chicago.

2 Algebraic and Real Vector Bundles

We start with the following theorem of Swan.

Theorem 2.1 (Swan) Let X be a compact connected Hausdorff space.

1. Let

$$C(X) = \{f : X \rightarrow \mathbb{R} \text{ is continuous function}\}.$$

2. Let $\pi : E \rightarrow X$ be a real vector bundle E on X . Let

$$\Gamma(E) = \{s : X \rightarrow E \text{ is a section of } E\}.$$

Note $\Gamma(E)$ is a projective $C(X)$ -module.

Then the association

$$E \longleftrightarrow \Gamma(E)$$

is an equivalence of categories. *The compactness and connectedness conditions can be relaxed.*

3 Non-vanishing sections

Today, we are interested in the existence of nowhere vanishing sections and obstructions. First, let me state Serre's theorem.

Theorem 3.1 Suppose A is a commutative noetherian ring of dimension n and P is a projective A -module with $\text{rank}(P) > n$. Then

$$P \approx Q \oplus A,$$

for some projective A -module Q . In other words P has a nowhere vanishing section.

Similarly, if V is a vector bundle over a compact manifold X with $\text{rank}(V) > \dim(X)$, then V has a nowhere vanishing section.

Question. So, what happens when $\text{rank}(P) = n = \dim(A)$? Note that the tangent bundle over the real two sphere does not have a nowhere vanishing section.

In topology, there is an obstruction theory ([MiS]) available to deal with similar questions for vector bundles over smooth compact manifolds.

A search for such an Obstruction theory in Algebra began with the work of Mohan Kumar and Murthy ([MK2, MKM, Mu1]) on vector bundles over affine algebras over algebraically closed fields. The final theorem is the following.

Theorem 3.2 (Murthy) Suppose A is reduced affine algebra of dimension n over an algebraically closed field k . Suppose $F^n K_0(A)$ had no $(n-1)!$ -torsion. Let P be a projective A -module of rank n . Then

$$P \approx Q \oplus A \iff C^n(P) = 0$$

where $C^n(P)$ denotes the top Chern class of P .

But, for projective A -modules P with $\text{rank}(P) = \dim(A) = n$, vanishing of the top Chern class $C^n(P) = 0$, is not a sufficient condition for existence of nowhere vanishing section. **The tangent bundle over real two sphere is an example.**

At this point, M. V. Nori introduced the Euler Class Group program ([MS, BS1] around 1989).

For an smooth affine variety $X = \text{Spec}(A)$ of dimension $n \geq 2$, he gave a definition of Euler class group $E(X)$ and he defined a Euler class $e(P) \in E(X)$ of vector bundles P of rank n over X with trivial determinant. He conjectured that

$$e(P) = 0 \iff P = Q \oplus A.$$

I did some work on this program of Nori ([Ma1, MS, MV]) and S. M. Bhatwadekar and Raja Sridharan ([BS1]) settled the conjecture affirmatively.

4 Main Definitions

For a commutative noetherian ring A of dimension $n \geq 2$ and a line bundle L on $\text{Spec}(A)$ a more general definition of relative Euler class group $E(A, L)$ and relative weak Euler class group $E_0(A, L)$ was given by Bhatwadekar and Sridharan [BS2].

Definition 4.1 (Skip to the short definition) Let A be a noetherian commutative ring with $\dim A = n$ and L be line bundle. Write $F = L \oplus A^{n-1}$.

1. For an ideal I of height n , two surjective homomorphisms $\omega_1, \omega_2 : F/IF \twoheadrightarrow I/I^2$ are said to be equivalent if $\omega_1\sigma = \omega_2$ for some automorphism $\sigma \in SL(F/IF)$. An equivalence class of surjective homomorphisms $\omega : F/IF \twoheadrightarrow I/I^2$ will be called a **local L -orientation**.

$$\begin{array}{ccc}
 F/IF & \xrightarrow{SL(F/IF)} & F/IF \\
 \searrow \omega_2 & & \swarrow \omega_1 \\
 & I/I^2 &
 \end{array}$$

2. Let

$$G(A, L) = \mathbb{Z} \langle \{(N, \omega) : N \text{ primary}, ht(N) = n\} \rangle$$

be the free abelian group generated by the set of all pairs (N, ω) (resp. by the set of all ideals N) where N is a primary ideal of height n and ω is a local L -orientation of N . Similarly, let

$$G_0(A) = \mathbb{Z} \langle \{(N) : N \text{ primary}, ht(N) = n\} \rangle .$$

3. Let J be an ideal of height n and

$$\omega : F/IF \twoheadrightarrow J/J^2$$

be a local L -orientation of J and

$$J = N_1 \cap N_2 \cap \cdots \cap N_k$$

an irredundant primary decomposition of J . Then

$$(J, \omega) := \sum_{i=1}^r (N_i, \omega_i) \in G(A, L)$$

denotes the cycle determined by (J, ω) . Also use

$$(J) := \sum_{i=1}^r (N_i) \in G_0(A).$$

4. A local L -orientation $\omega : F/IF \twoheadrightarrow I/I^2$ of an ideal I of height n is said to be a **global L -orientation**, if ω lifts to a surjection $\Theta : F \twoheadrightarrow I$.

$$\begin{array}{ccc} F & \xrightarrow{\Theta} & I \\ \downarrow & & \downarrow \\ F/IF & \xrightarrow{\omega} & I/I^2 \end{array}$$

5. Let

$$H(A, L) = \text{Subgroup}(\{(J, \omega) : \textit{it is GLOBAL}\}) \subseteq G(A, L)$$

and

$$H_0(A, L) = \text{Subgroup}(\{(J) : \textit{it is GLOBAL}\}) \subseteq G_0(L).$$

6. Define

$$E(A, L) := \frac{G(A, L)}{H(A, L)} \quad \text{and} \quad E_0(A, L) := \frac{G_0(A)}{H_0(A, L)}.$$

The group $E(A, L)$ is called the **Euler class group** of A (relative to L) and $E_0(A, L)$ is called the **weak Euler class group** of A (relative to L).

7. Now we assume that $\mathbb{Q} \subseteq A$. Given a projective A -module P with $\text{rank}(P) = n$ and an isomorphism (orientation) $\chi : L \xrightarrow{\sim} \wedge^n P$, we define **euler class** $e(P, \chi)$ as follows: Let

$$f : P \longrightarrow I$$

be a surjective homomorphism, where I is an ideal of height n . Now suppose $\gamma : F/IF \rightarrow P/IP$ is an isomorphism such that $(\wedge^n \gamma) = \overline{\chi}$ where "overline" denotes "modulo I ". Let $\omega = \overline{f}\gamma$.

$$\begin{array}{ccc} P & \xrightarrow{f} & I \\ \downarrow & & \downarrow \\ P/IP & \xrightarrow{\overline{f}} & I/I^2 \\ \uparrow \gamma & \nearrow \omega & \\ F/IF & & \end{array}$$

Define the **Euler class of** (P, χ) as

$$e(P, \chi) = (I, \omega) \in E(A, L).$$

Also define **weak Euler class of** P as

$$e_0(P) = (I) \in E_0(A, L).$$

Definition 4.2 (short definition) Let L be a line bundle over $\text{Spec}(A)$ Write $n = \dim(A)$ and $F = L \oplus A^{n-1}$

1. Let $G(A, L)$ be a free abelian group generated by the set of all (N, ω) where N is primary ideal with $\text{height}(N) = n$ and ω is an **equivalence class** of surjective maps

$$\begin{array}{ccc} F/IF & \xrightarrow{SL(F/IF)} & F/IF \\ & \searrow \omega_2 & \swarrow \omega_1 \\ & I/I^2 & \end{array}$$

2. Given a surjection map $\omega : F/IF \rightarrow I/I^2$, where I is any ideal with $\text{height}(I) = n$, it induces an element in

$$(I, \omega) \in G(A, L).$$

This is done by looking at the primary decomposition.

3. Let $H(A, L)$ be the subgroup $G(A, L)$ generated by **global** L -orientations (I, ω) , as defined by the diagram:

$$\begin{array}{ccc} F & \xrightarrow{\Theta} & I \\ \downarrow & & \downarrow \\ F/IF & \xrightarrow[\text{given}]{\omega} & I/I^2 \end{array}$$

Here $\omega : F/IF \rightarrow I/I^2$ is a surjective map and I is an ideal with $\text{height}(I) = n$.

4. Define **Euler class group**, as

$$E(A, L) = \frac{G(A, L)}{H(A, L)}.$$

5. Now we assume that $\mathbb{Q} \subseteq A$. Given a projective A -module P with

$$\text{rank}(P) = n \quad \text{and orientation} \quad \chi : L \xrightarrow{\sim} \wedge^n P,$$

we define the **Euler class of** (P, χ) as

$$e(P, \chi) = (I, \omega) \in E(A, L).$$

where (I, ω) is given by the diagram:

$$\begin{array}{ccc} P & \xrightarrow{f} & I \\ \downarrow & & \downarrow \\ P/IP & \xrightarrow{\bar{f}} & I/I^2 \\ \uparrow \scriptstyle \chi \sim \gamma & \nearrow \scriptstyle \omega & \\ F/IF & & \end{array}$$

6. Please compare the definitions with that of Chow group $CH_n(A)$ of zero cycles and the top Chern class $C^n(P)$.

The final result on vanishing conjecture of Nori is due to Bhatwadekar and Raja Sridharan ([BS2]).

Theorem 4.3 ([BS2]) Let A be a noetherian commutative ring of dimension $n \geq 2$, with $\mathbb{Q} \subseteq A$, and L be a line bundle on $\text{Spec}(A)$. Let P be an A -module of rank n and determinant L . Let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an orientation. Then

$$e(P, \chi) = 0 \iff P = Q \oplus A$$

for some A -module Q .

Bhatwadekar and Raja Sridharan ([BS2]) also proved:

Theorem 4.4 ([BS2]) Let A be a noetherian commutative ring of dimension $n \geq 2$, with $\mathbb{Q} \subseteq A$, and P be a projective A -module of rank n and $\det(P) = L$.

Suppose J is an ideal of height n and

$$\omega : (L \oplus A^{n-1})/J(L \oplus A^{n-1}) \twoheadrightarrow J/J^2$$

be a local L -orientation of J . Let $\chi : L \xrightarrow{\sim} \wedge^n P$ be an isomorphism and $e(P, \chi) = (J, \omega)$. Then there is a surjective map $\Theta : P \twoheadrightarrow J$, such that Θ and χ induces ω .

5 On Real Smooth affine Varieties

Recall, for tangent bundle T of the real two sphere S^2 , **the top Chern class $C_2(T) = 0$, but T does not have a nowhere vanishing section.** Still, Bhatwadekar and Raja Sridharan posed and initiated an investigation of the following question in [BS4].

Question. *Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over \mathbb{R} (the field of real numbers) and P a projective A -module of rank n .*

When does $C^n(P) = 0 \implies P \simeq A \oplus Q$?

They proved ([BS4]), if n is odd and $K_A \simeq A \simeq \wedge^n(P)$ then the vanishing of the top Chern class $C^n(P)$ is sufficient to conclude that $P \simeq A \oplus Q$, where $K_A = \wedge^n(\Omega_{A/\mathbb{R}})$ denotes the canonical module of A .

This question was settled in complete generality ([BDM]) as follows. Before we state the theorem, we introduce two notations.

Notation 5.1 Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field \mathbb{R} of real numbers.

1. Let $X(\mathbb{R})$ denote the smooth manifold of the real points in X . We always assume that $X(\mathbb{R}) \neq \emptyset$.
2. Let $\mathbb{R}(X) = S^{-1}A$ where S is the multiplicative set of all $f \in A$ so that f does not belong to any real maximal ideal. So, $\mathbb{R}(X)$ is gotten by **killing all the complex maximal ideals**. This ring represents the real manifold.

Theorem 5.1 Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field \mathbb{R} of real numbers. Let $K = \wedge^n(\Omega_{A/\mathbb{R}})$ denote the canonical module. Let P be a projective A -module of rank n and let $\wedge^n(P) = L$.

Assume that $C^n(P) = 0$ in $CH_0(X)$. Then

$$P \simeq A \oplus Q$$

in the following cases:

1. *Let $X(\mathbb{R})$ has no compact connected component.*
2. *For every compact connected component C of $X(\mathbb{R})$, $L_C \not\cong K_C$ where K_C and L_C denote restriction of (induced) line bundles on $X(\mathbb{R})$ to C .*
3. *n is odd.*

Moreover, if n is even and L is a rank 1 projective A -module such that there exists a compact connected component C of $X(\mathbb{R})$ with the property that $L_C \simeq K_C$, then there exists a projective A -module P of rank n such that $P \oplus A \simeq L \oplus A^{n-1} \oplus A$ (hence $C^n(P) = 0$) but P does not have a free summand of rank 1.

Note that this last part says that apart from the possible nonvanishing of its top Chern class, further

topological (i.e. non-algebraic) obstruction exists,

for an algebraic vector bundle of top rank over X to split off a trivial subbundle of rank 1.

The main thrust of our proof is in probing if

$$C^n(P) = 0 \implies e(P, \chi) = 0$$

for some orientation χ ? To do this, we need the structure theorems of Euler class groups and Chow groups.

6 Structure Theorem

Theorem 6.1 Let $X = \text{Spec}(A)$ be a smooth affine variety of dimension $n \geq 2$ over the field \mathbb{R} of real numbers and let $K = \wedge^n(\Omega_{A/\mathbb{R}})$ be the canonical module of A . Let L be a projective A -module of rank 1. Let $C_1, \dots, C_r, C_{r+1}, \dots, C_t$ be the compact connected components of $X(\mathbb{R})$ in the Euclidean topology. Let K_{C_i} and L_{C_i} denote restriction of (induced) line bundles on $X(\mathbb{R})$ to C_i . Assume that

$$L_{C_i} \simeq K_{C_i} \quad \text{for} \quad 1 \leq i \leq r$$

and

$$L_{C_i} \not\simeq K_{C_i} \quad \text{for} \quad r+1 \leq i \leq t.$$

Then,

$$E(\mathbb{R}(X), L) = \mathbb{Z}^r \oplus (\mathbb{Z}/(2))^{t-r}.$$

Proof of the Main Theorem 5.1: Let $L = \det(P)$. Fix an orientation $\chi : L \xrightarrow{\sim} \wedge^n P$. Let $e(P, \chi) \in E(A, L)$ be the euler class.

We have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccc}
& & 0 & & 0 & & \\
& & \downarrow & & \downarrow & & \\
& & K_1 & \xrightarrow{\sim} & K_2 & & \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E^{\mathbb{C}}(L) & \longrightarrow & E(A, L) & \longrightarrow & E(\mathbb{R}(X), L) = \mathbb{Z}^r \oplus \mathbb{Z}/(2)^{t-r} \longrightarrow 0 \\
& & \wr \downarrow \varphi & & \downarrow \Theta & & \downarrow \\
0 & \longrightarrow & CH(\mathbb{C}) & \longrightarrow & CH_0(A) & \longrightarrow & CH_0(\mathbb{R}(X)) = \mathbb{Z}/(2)^t \longrightarrow 0
\end{array}$$

From the structure theorem above and the knowledge of the group $CH_0(\mathbb{R}(X))$, due to Colliot-Thélène and Schiderer ([CT-S]), it follows that

$$K_2 \simeq 2\mathbb{Z}^r.$$

Case 1 : $X(\mathbb{R})$ has no compact connected component: In this case $E(\mathbb{R}(X), L) = 0$ and so $\Theta : E(A, L) \rightarrow CH_0(A)$ is an isomorphism. Since $C^n(P) = 0$, we have $e(P, \chi) = 0$. By theorem 4.3, $P \approx Q \oplus A$.

Case 2 : For every compact connected component C of $X(\mathbb{R})$ we have $L_C \not\cong K_C$.

So, $r = 0$ and $E(\mathbb{R}(X), L) = \mathbb{Z}/(2)^t$. By a THEOREM of Colliot-Thélène and Schiderer $\Theta : E(A, L) \rightarrow CH_0(A)$ is an isomorphism and theorem follows as above.

Case 3 : n odd: Let $\Delta : P \rightarrow P$ be multiplication by -1 . Then $\det(\Delta) = -1$. Let $\alpha : P \rightarrow I$ be a surjection where I is a (locally complete intersection) ideal of height n . Write $F = L \oplus A^{n-1}$. Using an isomorphism $\gamma : F/IF \xrightarrow{\sim} P/IP$, we get a $\omega : F/IF \rightarrow I/I^2$ so that

$$e(P, \chi) = (I, \omega) \in E(A, L).$$

Again $\alpha\Delta$ will induce $-\omega : F/IF \rightarrow I/I^2$ and hence

$$e(P, \chi) = (I, -\omega) \in E(A, L).$$

Therefore

$$2e(P, \chi) = (I, \omega) + (I, -\omega).$$

Also recall, $E_0(A, L) \approx CH_0(A)$. Since $C^n(P) = \text{cycle}(I) =$

0, we have class $[I] = 0 \in E_0(A, L)$. Therefore

$$2e(P, \chi) = (I, \omega) + (I, -\omega) = 0 \in E(A, L).$$

Also, $e(P, \chi) \in K_1$ the kernel of Θ . But K_1 is torsion free. Therefore $e(P, \chi) = 0$ and theorem follows.

Case 4: n even and for some compact component $L_C \simeq K_C$: It is not a news that $\Psi_L : E_0(A, L) \simeq CH_0(X)$. Also $\ker(\Theta)$ is free abelian group of rank at least one. So,

$$0 \rightarrow \ker(\Theta) \rightarrow E(A, L) \rightarrow E_0(A, L) \rightarrow 0$$

is exact.

From definition of $E_0(A, L)$ it follows that $\ker(\Theta)$ is generated by elements of the type (J, ω) where J is an ideal of height n and is image of $F = L \oplus A^{n-1}$.

Since $\ker(\Theta) \neq 0$, we pick a generator $(I, \omega) \neq 0 \in \ker(\Theta)$ such that F maps onto I .

Let $\alpha : F \rightarrow I$ be a surjective map and $\omega_0 : F/IF \rightarrow I/I^2$ be the induced orientation.

We can find f such that the diagram

$$\begin{array}{ccc} F/IF & \xrightarrow{\omega} & I/I^2 \\ \downarrow f & & \parallel \\ F/IF & \xrightarrow{\omega_0} & I/I^2 \end{array}$$

commutes. Note that $\det(f) = \bar{u}$ has to be a unit (check mod maximal ideals and use height). So, $\omega = \bar{u}\omega_0$ and

$\bar{u}\bar{v} = 1$. Let $M = \ker(\alpha)$, and we have the exact sequence:

$$0 \rightarrow M \rightarrow F \xrightarrow{\alpha} I \rightarrow 0.$$

We can consider this exact sequence as an element of $z \in \text{Ext}(I, M)$. Then vz is given by the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & F & \xrightarrow{\alpha} & I \longrightarrow 0 \\ & & \downarrow v & & \downarrow g & & \parallel \\ 0 & \longrightarrow & M & \longrightarrow & P & \xrightarrow{\beta} & I \longrightarrow 0 \end{array}$$

Therefore P is projective and $[P] = [F]$. Also

$$e(P, \chi) = (I, \omega) \neq 0$$

for some orientation $\chi : L \xrightarrow{\sim} \wedge^n P$.

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