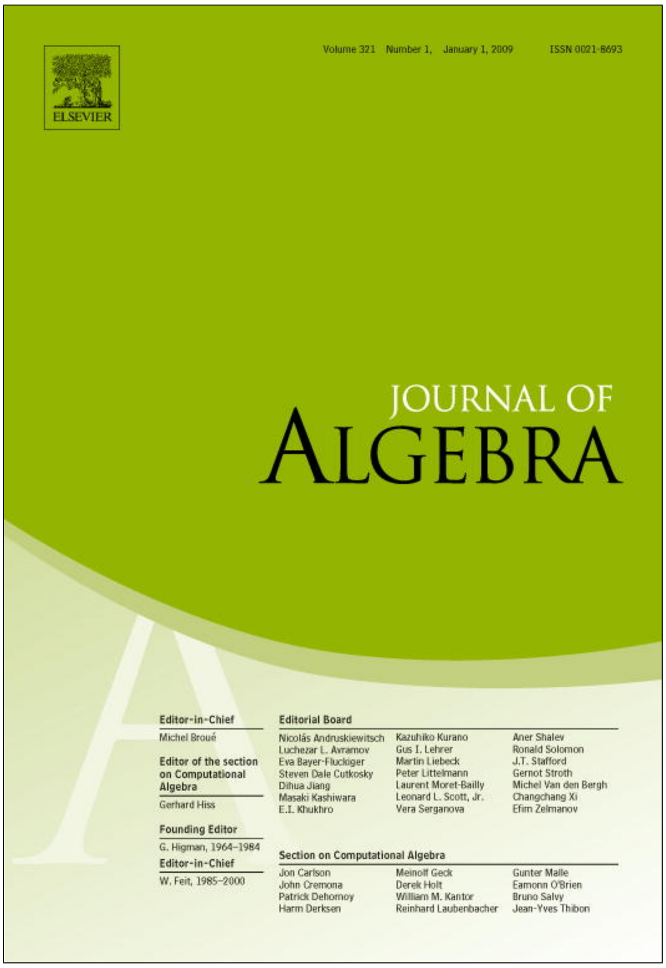
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# Bott periodicity and calculus of Euler classes on spheres

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ABSTRACT

A variety of computations regarding the Euler class group $E(A_n, A_n)$ and the Grothendieck group $K_0(A_n)$ of the algebraic sphere $Spec(A_n)$ is done. The Euler class of the algebraic tangent bundle on $Spec(A_n)$ is computed. It is also investigated whether every element in the Euler class group $E(A_n, A_n)$ is the Euler class of a projective $A_n$ module of rank $n$.



## 1. Introduction

Work on obstruction theory for projective modules started with the work of N. Mohan Kumar and M.P. Murthy [Mk,MkM,Mu1]. It is a result of Murthy [Mu1] that *for a reduced (smooth) affine algebra $A$ with* $\dim A = n$, *over an algebraically closed field k, the top Chern class map* $C_0 : K_0(A) \to CH_0(A)$ *is surjective*. This result is a consequence of the result [Mu1] that *given any local complete intersection ideal $I$ of height n, there is a projective $A$-module $P$ with* $\mathrm{rank}(P) = n$ *that maps surjectively onto $I$*.

For real smooth affine varieties such propositions will fail. Most common examples are that of real spheres. We denote the real sphere of dimension $n$ by $\mathbb{S}^n$ and $A_n$ denotes the ring of algebraic functions on $\mathbb{S}^n$. We have, the Chow group of zero cycles $CH_0(A_n) = \mathbb{Z}/2$ (see 3.1) and by the theorem of Swan [Sw2], $K_0(A_n) = KO(\mathbb{S}^n)$. By the periodicity theorem of Bott (see 5.10), for nonnegative integers $n = 8r + 3, 8r + 5, 8r + 6, 8r + 7$ $(r \geqslant 0)$ we have $K_0(A_n) = KO(\mathbb{S}^n) = \mathbb{Z}$. In these cases, the top Chern class map $C_0 = 0$ and it fails to be surjective.

On the other hand, by Bott periodicity (see 5.10), $\widetilde{K_0}(A_{8r}) = \widetilde{KO}(\mathbb{S}^{8r}) = \mathbb{Z}$, $\widetilde{K_0}(A_{8r+1}) = \widetilde{KO}(\mathbb{S}^{8r+1}) = \mathbb{Z}/(2)$, $\widetilde{K_0}(A_{8r+2}) = \widetilde{KO}(\mathbb{S}^{8r+2}) = \mathbb{Z}/(2)$, $\widetilde{K_0}(A_{8r+4}) = \widetilde{KO}(\mathbb{S}^{8r+4}) = \mathbb{Z}$. Therefore, in these cases, the question of surjectivity of the top Chern class map $C_0 : K_0(A_n) \to CH_0(A_n)$ fully depends on the top Chern class of the generator $\tau_n$ of $\widetilde{K_0}(A_n)$. In analogy to the obstruction theory in topology, it makes

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more sense to consider the Euler class group $E(A_n)$ of $A_n$ as the obstruction group, instead of the Chow group $CH_0(A_n)$.

For (smooth) affine rings $A$ with $\dim A = n \geqslant 2$, over a field $k$, the original definition of Euler class groups $E(A)$ was given by Nori [MS,BRS2]. For a projective $A$-module $P$, with $\det P = A$ and an orientation $\chi : A \xrightarrow{\sim} \det P$, an Euler class $e(P, \chi) \in E(A)$ was defined. We mainly refer to [BRS2], for basics on Euler class groups and Euler classes. For such a ring $A$, $\mathcal{P}O_n(A)$ will denote the set of all isomorphism classes of pairs $(P, \chi)$, where $P$ is a projective $A$-module rank $n$, with trivial determinant, and $\chi : A \xrightarrow{\sim} \det P$ is an isomorphism, to be called an orientation.

So, our main question is *whether the Euler class map $e : \mathcal{P}O_n(A_n) \to E(A_n)$ is surjective*. In fact, $E(A_n) = \mathbb{Z}$. For reasons given above, the Euler class map fails to be surjective for $n = 8r + 3, 8r + 5, 8r + 6, 8r + 7$ $(r \geqslant 0)$. *In fact, we also prove that for $n = 8r + 1$ this map fails to be surjective. For any even integer $n \geqslant 2$, we prove that any even class $N \in E(A_n) = \mathbb{Z}$, is in the image of $e$. For $n = 2, 4, 8$ we prove that $e$ is surjective. For $n = 8r, 8r + 2, 8r + 4 \geqslant 2$, we prove that $e$ is surjective if and only if the top Stiefel–Whitney class $w_n(\tau_n) = 1$ where $\tau_n$ is the generator of $\widetilde{K}_0(A_n)$*. It remains an open question whether $w_n(\tau_n) = 1$. It follows (see 6.4) that $w_n(\tau_n) = 1$ if an only if $C_0(\tau_n) = 1$.

Among other results in this paper, we compute (see 3.3) the Euler class of the algebraic tangent bundle $T$ over $Spec(A_n)$. As in topology (see [MiS]), $e(T, \chi) = -2$, when $n$ is even and zero when $n$ is odd. This provides a fully algebraic proof that the algebraic tangent bundles $T$ over even dimensional spheres $Spec(A_n)$ do not have a free direct summand.

Given any real maximal ideal $m$ of $A_n$, we attach (see 4.2) a local orientation $\omega$ on $m$ in an algorithmic way and compute the class $(m, \omega) = 1$ or $(m, \omega) = -1$ in $E(A_n)$.

## 2. Preliminaries

Following are some of the notations we will be using in this paper.

**Notations 2.1.** First, the fields of real numbers and complex numbers will, respectively, be denoted by $\mathbb{R}$ and $\mathbb{C}$. The quaternion algebra will be denoted by $\mathbb{H}$.

1. The real sphere of dimension $n$ will be denoted by $\mathbb{S}^n$. Let
$$A_n = \frac{\mathbb{R}[X_0, X_1, \dots, X_n]}{(\sum_{i=0}^n X_i^2 - 1)} = \mathbb{R}[x_0, x_1, \dots, x_n]$$
denote the ring of algebraic functions on $\mathbb{S}^n$.
2. For any real affine variety $X = Spec(A)$, let $\mathbb{R}(X) = S^{-1}A$, where $S$ is the multiplicative set of all $f \in A$ that do not vanish at any real point of $Spec(A)$. Also, $X(\mathbb{R})$ denote the set of all real points of $X$.
3. For any noetherian commutative ring $A$ and line bundles $L$ on $Spec(A)$, the Euler class group will be denoted by $E(A, L)$ and the weak Euler class group will be denoted by $E_0(A, L)$. Usually, $E(A, A)$ will be denoted by $E(A)$ and similarly $E_0(A)$ will denote $E_0(A, A)$. We refer to [BRS2] for the definitions and the basic properties of these groups.

The following theorem would be obvious to the experts (see [BRS1]).

**Theorem 2.2.** *Let $X = spec(A)$ be a smooth affine variety of dimension $n \geqslant 2$ over $\mathbb{R}$. Then, the natural map*
$$E_0\big(\mathbb{R}(X)\big) \to CH_0\big(\mathbb{R}(X)\big)$$
*is an isomorphism and $CH_0(\mathbb{R}(X)) \approx \mathbb{Z}/(2)^r$ where $r$ is the number of compact connected components of $X(\mathbb{R})$.*

**Proof.** It follows directly from [BRS1, Theorem 5.5] and Theorem 2.3 below that $E_0(\mathbb{R}(X)) \xrightarrow{\sim} CH_0(\mathbb{R}(X))$. Also, by [BRS1, Theorem 4.10], $CH_0(\mathbb{R}(X)) \approx \mathbb{Z}/(2)^r$. $\square$

**Theorem 2.3.** *(See [BDM].) Let $X = spec(A)$ be smooth affine variety of dimension $n \geqslant 2$ over $\mathbb{R}$. The following diagram of exact sequences*

$$\begin{tikzcd} 0 \arrow[r] & E^{\mathbb{C}}(L) \arrow[r] \arrow[d, "\varphi", "\wr"'] & E(A, L) \arrow[r] \arrow[d, "\Theta"] & E(\mathbb{R}(X), L) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & CH(\mathbb{C}) \arrow[r] & CH_0(A) \arrow[r] & CH_0(\mathbb{R}(X)) \arrow[r] & 0 \end{tikzcd}$$

*commute and the first vertical map $\varphi$ is an isomorphism.*

**Proof.** We only need to prove that $\varphi$ is injective. The proof is given in the proof of [BDM, Proposition 4.29]. $\square$

We also include the following easy lemma.

**Lemma 2.4.** *Let $A$ be any smooth affine ring over $\mathbb{R}$ with $\dim A = n \geqslant 2$ and $L$ be a line bundle on Spec$(A)$. Let $P$ be a projective $A$-module of rank $n$ and $\det P = L$. Let $\chi, \eta : L \xrightarrow{\sim} \bigwedge^n P$ be two orientations. Suppose $e(P, \chi) = (I, \omega)$ where $I$ is an ideal of height $n$ and $\omega$ is a local orientation on $I$ and $\eta = u\chi$ where $u$ is a unit in $A$. Then $e(P, \eta) = (I, u\omega)$.*

**Proof.** Write $F = L \oplus A^{n-1}$. By theorem in [BRS2], there is a surjective map $f : F \to I$ that induces $(I, \omega)$ as in the commutative diagram:

$$\begin{tikzcd} P \arrow[r] \arrow[d, "f"'] & P/IP \arrow[d] & F/IF \arrow[l, "\gamma\sim\chi"', "\sim"] \arrow[dl, "\omega"] \\ I \arrow[r] & I/I^2 & F/IF. \arrow[l, "\omega\delta"] \arrow[u, dashed, "\delta"'] \end{tikzcd}$$

Here $\gamma$ is an isomorphism with determinant $\chi$, and $\delta$ is any isomorphism with $\det(\delta) = u$. So $\gamma\delta \sim u\chi = \eta$ and $e(P, \eta) = (I, u\omega)$. $\square$

## 3. The tangent bundle

It is well known that the tangent bundle $T_n$, over the real sphere $\mathbb{S}^n$, of even dimension $n \geqslant 1$, does not have a nowhere vanishing section. The purpose of this section is to compute the Euler class of the algebraic tangent bundle explicitly.

First, note that all line bundles over $\mathbb{S}^n$, with $n \geqslant 2$ are trivial, we have only one Euler class group $E(A_n, A_n)$ to be denoted by $E(A_n)$. Similarly, we have only one weak Euler class group $E_0(A_n)$. The following proposition entails some of the basic facts about Euler class groups of the spheres.

**Proposition 3.1.** *The Euler class group of the sphere is given by $E(A_n) = \mathbb{Z}$, generated by $(m, \omega)$ where $m$ is any real maximal ideal and $\omega$ is any local orientation of $m$. Similarly, the weak Euler class group is given by*

$$E_0(A_n) \approx CH_0(A_n) = \frac{\mathbb{Z}}{(2)}.$$

**Proof.** From Theorem 2.3, we have the commutative diagram of exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & E^{\mathbb{C}}(A_n) \arrow[r] \arrow[d, "\varphi", "\wr"'] & E(A_n) \arrow[r] \arrow[d, "\Theta"] & E(\mathbb{R}(\mathbb{S}^n)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & CH(\mathbb{C}) \arrow[r] & CH_0(A_n) \arrow[r] & CH_0(\mathbb{R}(\mathbb{S}^n)) \arrow[r] & 0.
\end{tikzcd}$$

Since complex points in $A_n$ are complete intersection [MS, Lemma 4.2], we have $CH(\mathbb{C}) = E^{\mathbb{C}}(A_n) = 0$ and the above diagram reduces to

$$\begin{tikzcd}
0 \arrow[r] & E(A_n) \arrow[r, "\sim"] \arrow[d, "\Theta"] & E(\mathbb{R}(\mathbb{S}^n)) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & CH_0(A_n) \arrow[r, "\sim"] & CH_0(\mathbb{R}(\mathbb{S}^n)) \arrow[r] & 0.
\end{tikzcd}$$

We have by Theorem 2.2, $E_0(A_n) \xrightarrow{\sim} CH_0(A_n)$. Therefore, by [BRS1, Theorems 4.13, 4.10]

$$E\big(\mathbb{R}\big(\mathbb{S}^n\big)\big) = \mathbb{Z} \quad \text{and} \quad CH_0(A_n) \approx E_0\big(\mathbb{R}\big(\mathbb{S}^n\big)\big) = \mathbb{Z}/(2).$$

The proof is complete. $\square$

The following definition will be convenient for subsequent discussions.

**Definition 3.2.** Let $m_0 = (x_0 - 1, x_1, \dots, x_n)$ be the maximal ideal in $A_n$ that corresponds to the real point $(1, 0, \dots, 0) \in \mathbb{S}^n$. Write $F = A_n^n$ and let $e_1, \dots, e_n$ be the standard basis of $F$. Define local orientation

$$\omega_0 : F/m_0F \to m_0/m_0^2 \quad \text{where for } i = 1, \dots, n, \quad \omega(e_i) = image(x_i).$$

By Proposition 3.1, $(m_0, \omega_0)$ will generate the Euler class group $E(A_n) = \mathbb{Z}$. This generator $(m_0, \omega_0) = 1$ will be called the **standard generator** of $E(A_n)$. Similarly, the class of $m_0 = 1$ will be called the **standard generator** of $E_0(A_n) = \mathbb{Z}/(2)$.

Unless stated otherwise, we use these standard generators in our subsequent discussions.

We compute the Euler class of the algebraic tangent bundle over $Spec(A_n)$ as follows.

**Theorem 3.3.** *Let $T_n$ be the projective $A_n$-module corresponding to the tangent bundle over $\mathbb{S}^n$. There is an orientation $\chi : A_n \xrightarrow{\sim} \bigwedge^n T_n$ such that, if $n \geqslant 2$ is even, then the Euler class $e(T_n, \chi) = -2 \in E(A_n)$ and if $n \geqslant 3$ is odd, then the Euler class $e(T_n, \chi) = 0 \in E(A_n)$.*

**Proof.** Write $m_0 = (x_0 - 1, x_1, \dots, x_n)$, $m_1 = (x_0 + 1, x_1, \dots, x_n) \in Spec(A_n)$. Then $m_0$, $m_1$ correspond, respectively, to the points $(1, 0, \dots, 0)$, $(-1, 0, \dots, 0)$ in $\mathbb{S}^n$. We have

$$m_0 = (x_1, \dots, x_n) + m_0^2, \qquad m_1 = (x_1, \dots, x_n) + m_1^2,$$

and $m_0 \cap m_1 = (x_1, \dots, x_n)$. Write $F = A_n^n$ and let $e_1, \dots, e_n$ be the standard basis. For $j = 0, 1$ we define local orientations

$$\omega_j : F/m_jF \to m_j/m_j^2 \quad \text{where for } i = 1, \dots, n, \quad \omega_j(e_i) = image(x_i).$$

Therefore, $(m_0,\omega_0)=1$ is the standard generator of $E(A_n)=\mathbb{Z}$. We write $J=m_0\cap m_1=(x_1,\ldots,x_n)$ and define the surjective map

$$\alpha : F \twoheadrightarrow J \quad \text{where for } i\geqslant 1, \quad \alpha(e_i)=x_i.$$

Then, $\alpha$ induces the local orientation

$$\omega : F/JF \to J/J^2 \quad \text{where } \omega(e_i)=image(x_i).$$

Since $(J,\omega)$ is global, it follows

$$(m_0,\omega_0)+(m_1,\omega_1)=(J,\omega)=0.$$

Hence

$$(m_1,\omega_1)=-(m_0,\omega_0)=-1.$$

Since $E(A_n)=E(\mathbb{R}(\mathbb{S}^n))$, we can apply [BDM, Lemma 4.2] and we have

$$(m_1,\omega_1)+(m_1,-\omega_1)=0.$$

Therefore

$$(m_0,\omega_0)+(m_1,-\omega_1)=2(m_0,\omega_0)=2.$$

Let $D=diagonal(-x_0,1,\ldots,1) : F/JF\to F/JF$, then $D$ is an automorphism and $\det(D)=image(-x_0)$. Now, let $\eta=\omega D : F/JF\to J/J^2$. In fact,

$$\eta(e_1)=image(-x_0x_1) \quad \text{and} \quad \eta(e_i)=image(x_i) \quad \forall i>1.$$

Note that

$$D=diagonal(-1,1,\ldots,1) \mod m_0, \qquad D=Id \mod m_1.$$

Since, $\omega_i$ are the reductions of $\omega$ modulo $m_i$ we have

$$(J,\eta)=(m_0,-\omega_0)+(m_1,\omega_1)=-\big[(m_0,\omega_0)+(m_1,-\omega_1)\big]=-2.$$

Now we apply [BRS2, Lemma 5.1], to $\alpha : F\twoheadrightarrow J$, with $a=b=image(-x_0)$. We have the following:

1. Define $T$ by the exact sequence

$$0\to T\to A_n\oplus F=A^{n+1}\xrightarrow{\Phi} A_n\to 0$$

where

$$\Phi=-(x_0,x_1,\ldots,x_n)=(b,-\alpha).$$

2. We have $(J,\omega)$ is obtained from $(\alpha,\chi_0=Id_{A_n})$.
3. By [BRS2, Lemma 5.1], $T$ has an orientation $\chi : A_n\to\bigwedge^n T$ such that

$$e(T,\chi)=\big(J, image(-x_0)^{n-1}\omega\big)=\big(m_0,(-1)^{n-1}\omega_0\big)+(m_1,\omega_1).$$

4. If $n$ is **EVEN**, we have

$$e(T,\chi) = (m_0, -\omega_0) + (m_1, \omega_1) = -2.$$

And if, $n$ is **ODD**, we have

$$e(T,\chi) = (m_0, \omega_0) + (m_1, \omega_1) = 0.$$

5. Note that $T = \ker(\Phi) \approx \ker(-\Phi) = T_n$ is the tangent bundle.

So, the proof is complete. □

## 4. An algorithmic computation in $E(A_n)$

**Lemma 4.1.** *Let $A_n$ be as above and let $m_1, M_1, m_2, M_2, \dots, m_N, M_N \in spec(A_n)$ be a set of distinct maximal ideals that correspond to distinct real points in $\mathbb{S}^n$. We will assume that these points are in $\mathbb{S}^1 = \{x_j = 0: \forall j \geqslant 2\} \subseteq \mathbb{S}^n$. For $i = 1, \dots, N$, let $L_i = 0, x_2 = 0, \dots, x_n = 0$ be the line passing through the pair of points corresponding to $m_i$ and $M_i$. Then*

$$\bigcap_{i=1}^{N} (m_i \cap M_i) = \left( \prod_{i=1}^{N} L_i, x_2, \dots, x_n \right).$$

**Proof.** Let $J$ denote the right-hand side. Claim that

$$J \subseteq m \in Spec(A_n) \quad \Rightarrow \quad m = m_i \quad \text{or} \quad m = M_i \quad \text{for some } i.$$

To see this, note for such an $m$, we have $L_i \in m$ for some $i$. Therefore,

$$m_i \cap M_i = (L_i, x_2, \dots, x_n) \subseteq m.$$

Hence $m = m_i$ or $M_i$. Let $m$ be such a maximal ideal and assume $m = m_i$. We have, $L_j \notin m_i\ \forall j \neq i$ and $J_{m_i} = (L_i, x_2, \dots, x_n)_{m_i} = (m_i)_{m_i}$. The proof is complete. □

Given various points $m$ in $\mathbb{S}^n$, the following is an algorithm to compute class $(m, \omega) \in E(A_n)$.

**Theorem 4.2.** *As in Definition 3.2, let $(m_0, \omega_0) = 1 \in E(A_n) = \mathbb{Z}$ be the standard generator. Let $p = (a, b, 0, \dots, 0)$ be a point in $\mathbb{S}^n$ and let $M = (x_0 - a, x_1 - b, x_2, \dots, x_n) \in Spec(A_n)$ be the maximal ideal corresponding to $p$. Assume $m_0 \neq M$ and so $a \neq 1$. Let*

$$L = (1-a)x_1 + b(x_0 - 1), \quad \textit{so} \quad (L, x_2, x_3, \dots, x_n) = m_0 \cap M.$$

*As in* (3.2), *$F = A_n^n$ and $e_1, \dots, e_n$ is the standard basis of $F$. Define*

$$\omega_M : F/MF \to M/M^2 \quad \textit{by}\ \omega_M(e_1) = x_1 - b, \quad \omega_M(e_i) = x_i \quad \forall i \geqslant 2.$$

*If $a \neq 0$* (i.e. $p$ is not the north or the south pole), *then $\omega_M$ is a surjective map and*

$$(m_0, \omega_0) + \big(M, -sign(a)\omega_M\big) = 0.$$

*So, if $a > 0$ then $(M, \omega_M) = 1$ and $a < 0$ then $(M, \omega_M) = -1$.*

**Proof.** Define the surjective map

$$f : F \to m_0 \cap M \quad \text{by } f(e_1) = L, \quad f(e_i) = x_i \quad \forall i \geqslant 2.$$

We will see that $f$ reduces to $\omega_0$ modulo $m_0$. With $s = -1/2, t = 1/2$ we have, $1 = s(x_0 - 1) + t(x_0 + 1)$. So

$$(x_0 - 1) = s(x_0 - 1)^2 + t\bigl(x_0^2 - 1\bigr) = s(x_0 - 1)^2 + t\sum_{i=1}^{n} -x_i^2 \in m_0^2.$$

Therefore $L - (1 - a)x_1 \in m_0^2$. Also since $M \neq m_0$ we have $a \neq 1$. In fact $a < 1$. Hence $f$ reduces to

$$\omega_0 : F/m_0F \to m_0/m_0^2.$$

Now define

$$\gamma_M : F/MF \to M/M^2 \quad \text{by } \gamma_M(e_1) = L, \quad \gamma_M(e_i) = x_i \quad \forall i \geqslant 2.$$

Since $\gamma_M$ is the reduction of $f$, we have

$$(m_0, \omega_0) + (M, \gamma_M) = 0.$$

Let $\omega_M : F/MF \to M/M^2$ be as in the statement of the theorem. We will assume $a \neq 0$ or equivalently, $-1 < b < 1$. In this case, we prove that $\omega_M$ is a surjective map. (*Note below that for $\omega_M$ to be surjective, we need $a \neq 0$.*) We have

$$L = (1 - a)x_1 + b(x_0 - 1) = (1 - a)(x_1 - b) + b(x_0 - a).$$

We also have $a^2 + b^2 - 1 = 0$. Now again, with $s = -1/2a, t = 1/2a$ we have $1 = s(x_0 - a) + t(x_0 + a)$ and

$$(x_0 - a) = s(x_0 - a)^2 + t\bigl(x_0^2 - a^2\bigr) = s(x_0 - a)^2 + t\bigl(b^2 - x_1^2\bigr) - t\sum_{i=2}^{n} x_i^2.$$

Therefore, $\omega_M$ is surjective. Further,

$$\bigl(b^2 - x_1^2\bigr) = (b - x_1)(b + x_1) = (b - x_1)\bigl[2b - (b - x_1)\bigr] = 2b(b - x_1) - (b - x_1)^2.$$

So,

$$(x_0 - a) = s(x_0 - a)^2 + t\bigl[2b(b - x_1) - (b - x_1)^2\bigr] - t\sum_{i=2}^{n} x_i^2 = b/a(b - x_1) + w$$

for some $w \in M^2$. So,

$$(x_0 - a) - b/a(b - x_1) \in M^2.$$

Therefore, *modulo* $M$, we have

$$L = (1 - a)(x_1 - b) + b(x_0 - a) \equiv (1 - a)(x_1 - b) + b(b/a)(b - x_1)$$

or

$$L \equiv (x_1 - b)\big[1 - a - b^2/a\big] = (x_1 - b)\big[(a-1)/a\big].$$

So, $\gamma_M$ and $\omega_M$ differ by an isomorphism of determinant $(a-1)/a$. Since $a - 1 < 0$, we have $\gamma_M = -sign(a)\omega_M$. Therefore,

$$(m_0, \omega_0) + \big(M, -sign(a)\omega_M\big) = 0.$$

Hence, if

$$a > 0 \quad \Rightarrow \quad (M, \omega_M) = -(M, -\omega_M) = (m_0, \omega_0) = 1$$

and

$$a < 0 \quad \Rightarrow \quad (M, \omega_M) = -(m_0, \omega_0) = -1.$$

So, the proof is complete. $\square$

**Remark 4.3.** We will continue to use the notations of (3.2, 4.2). As we remarked in the proof of Theorem 4.2, if $p$ is the north pole or the south pole and $M$ is the corresponding maximal ideal, then $\omega_M$, as defined in 4.2, will fail to define a local orientation. If $p = (0, \pm 1, 0, \ldots, 0)$ is the north or the south pole, then $M = (x_0, x_1 \mp 1, x_2, \ldots, x_n)$. For $p = N$ the north pole or $p = S$ the south pole, a natural local orientation is defined by:

$$\omega_p : F/MF \to M/M^2 \quad \text{where } \omega_p(e_1) = x_0, \quad \omega_p(e_i) = x_i \quad \forall i \geqslant 2.$$

Then $(M, \omega_p) = -1$ if $p$ is the north pole and $(M, \omega_p) = 1$ if $p$ is the south pole.

**Proof.** Let $p = N = (0, 1, 0, \ldots, 0)$ be the north pole. Then $M = (x_0, x_1 - 1, x_2, \ldots, x_n)$. Write $L = x_0 + x_1 - 1$. Then $m_0 \cap M = (L, x_2, \ldots, x_n)$. Consider the surjective map $f : F \twoheadrightarrow m_0 \cap M$ given by these generators. Note that $L - x_0 = x_1 - 1 \in M^2$. This follows because $1 = -(x_1 - 1)/2 + (x_1 + 1)/2$. So, it follows that $f$ reduces to $\omega_p$. Similarly, $f$ reduces to $\omega_0$ on $m_0$. Therefore, $(M, \omega_p) = -1$.

If $p = S = (0, -1, 0, \ldots, 0)$ is the south pole, then $M = (x_0, x_1 + 1, x_2, \ldots, x_n)$. We replace the equation of $L$ by $L = x_0 - x_1 - 1$. Then $L - x_0 = -(x_1 + 1) \in M^2$. It follows, that $f$ reduced to $\omega_p$. Similarly, $L + x_1 = x_0 - 1 \in m_0^2$. This shows that $f$ reduces to $-\omega_0$ on $m_0$. Therefore, $(M, \omega_p) - (m_0, \omega_0) = 0$. So, $(M, \omega_p) = 1$. The proof is complete. $\square$

**Remark 4.4.** In the statement of (4.2), we assumed that $p = (a, b, 0, \ldots, 0) \in \mathbb{S}^1 \subseteq \mathbb{S}^n$. Now suppose $p \notin \mathbb{S}^1$ is any point in $\mathbb{S}^n$. Let $e_0, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. So, $m_0$ is the ideal of $e_0$. There is an orthonormal transformation $(E_0, \ldots, E_n)^t = A(e_0, \ldots, e_n)^t$ of $\mathbb{R}^{n+1}$ such that $E_0 = e_0$ and $p = aE_0 + bE_1$. Write $A = (a_{ij}\colon\ i, j = 0, \ldots, n)$. It follows, $a_{00} = 1$ and $a_{0j} = a_{j0} = 0$ for all $j = 1, \ldots, n$. We can assume $\det A = 1$.

Write $(Y_0, \ldots, Y_n)^t = A(X_0, \ldots, X_n)^t$. Then $Y_0 = X_0$, and for $i = 1, \ldots, n$ we have $Y_i = \sum_{j=1}^n a_{ij} X_j$. It follows that in the $Y$-coordinates, $e_0 = (1, 0, \ldots 0)$ and $p = (a, b, 0, \ldots, 0)$. Let $\omega'$ be the local orientation on $m_0$ defined by $(Y_1, \ldots, Y_n)$. Since $\det A = 1$, it follows that $(m_0, \omega')$ is the standard generator of $E(A_n)$ (see 3.2). Now, we can write down local orientation on $M$ in $Y$-coordinates, as in (4.2) and the rest of (4.2) remains valid.

## 5. Bott periodicity

In this section, we will give some background on Bott periodicity, mostly from [ABS,F,Sw1]. We will recall the definition of the Clifford algebras of a quadratic forms.

**Definition 5.1.** (See [ABS].) Let $k$ be a commutative ring and $(V,q)$ be a quadratic $k$-module. Then a $k$-algebra $C(q)$ with an injective map $i: V \to C(q)$ is said to be the **Clifford algebra** of $q$, if $i(x)^2 = q(x)$ and if it is universal with respect to this property. *Following are some of the properties of* $C(q)$:

1. Note $C(q) = \frac{T(V)}{I(q)}$ where $T(V)$ is the tensor algebra of $V$ and $I(q)$ is the two-sided ideal of $T(V)$ generated by $\{x^2 - q(x)\colon\ x \in V\}$.
2. The $\mathbb{Z}_2$-grading on $T(V)$ induces a $\mathbb{Z}_2$-grading on $C(q)$ as $C(q) = C_0(q) \oplus C_1(q)$ where $C_0(q)$ denotes the even part and $C_1(q)$ denotes the odd part.
3. Also, if $(V, q')$ is another quadratic $k$-module, then
$$C(q \perp q') \approx C(q) \mathbin{\widehat{\otimes}} C(q') \quad \text{as graded rings.}$$
This means, the multiplication structure is given by $(u \otimes x_i)(y_j \otimes v) = (-1)^{ij} u y_j \otimes x_i v$ for $x_i \in C_i(q')$, $y_j \in C_j(q)$.
4. If $V = \bigoplus_{i=1}^n ke_i$ is free with basis $e_i$ then
$$C(q) = \bigoplus_{0 \leqslant i_1 < \cdots < i_r \leqslant n;\ r \geqslant 0} ke_{i_1 i_2 \ldots i_r}.$$
We will mostly be concerned with this case where $V$ is free. Further, if $q = \sum_{i=1}^n a_i X_i^2$ is a diagonal form, then
$$\forall i, j = 1, \ldots, n; \quad \text{with } i \neq j, \quad e_i^2 = a_i \quad \text{and} \quad e_i e_j = -e_j e_i.$$

**Notations 5.2.** We will introduce some notations for our convenience.

1. Let $k$ be a commutative ring and $(V,q)$ be a quadratic $k$-module and $V = \bigoplus_{i=1}^n ke_i$ is free and $q = q(X_1, \ldots, X_n)$. As in [Sw1], we denote $R_k(q) = R(q) = \frac{k[X_1,\ldots,X_n]}{(q-1)}$. We usually drop the subscript $k$ and use the notation $R(q)$.
2. Suppose $C$ is a ring. Then:
   (a) The category of finitely generated (left) $C$-modules will be denoted by $\mathcal{M}(C)$.
   (b) If $C$ has a $\mathbb{Z}_2$-grading, the category of finitely generated (left) $\mathbb{Z}_2$-graded $C$-modules will be denoted by $\mathcal{G}(C)$.
   (c) The category of finitely generated (left) projective $C$-modules will be denoted by $\mathcal{P}(C)$.
3. Given a category $\mathcal{C}$, with exact sequences, the Grothendieck group of $\mathcal{C}$ will be denoted $K(\mathcal{C})$.
4. Given a ring $R$, we will denote $K_0(R) = K(\mathcal{P}(R))$. If the rank map $rank: K_0(R) \to \mathbb{Z}$ is defined, we denote $\widetilde{K_0}(R) = rank^{-1}(0)$.
5. Given a connected smooth real manifold $X$, the Grothendieck group of the category of real vector bundles over $X$ will be denoted by $KO(X)$. As above, $\widetilde{KO}(X)$ will denote the kernel of the rank map.
6. For a commutative noetherian ring $R$ of dimension $n$ and $X = Spec(R)$, the Chow group of zero cycles will be denoted by $CH_0(R)$ or $CH_0(X)$. When the top Chern class is defined, $C_0 = C^n: K_0(R) \to CH_0(R)$ will denote the homomorphism defined by the top Chern class.

### 5.1. Generators of $\widetilde{K_0}(A_n)$

In this subsection, we describe the generators of $\widetilde{K_0}(A_n)$.

**Proposition 5.3.** *Let $k$ be ring with $1/2 \in k$ and let $q = a_1X_1^2 + \cdots + a_nX_n^2$ be a diagonal form. Let $e_1, \ldots, e_n$ denote the canonical generators of $C(-q)$. Let $M = M^0 \oplus M^1 \in \mathcal{G}(C(-q))$ be a $\mathbb{Z}_2$-graded $C(-q)$-module and*

$$N = R(q \perp 1) \otimes_k M = N^0 \oplus N^1$$

*where*

$$N^0 = R(q \perp 1) \otimes_k M^0, \qquad N^1 = R(q \perp 1) \otimes_k M^1.$$

*Let $x_i$ denote the image of $X_i$ in $R(q \perp 1)$. Define*

$$\varphi(x) = \sum_{i=1}^{n} x_i(1 \otimes e_i) : N^1 \to N^0, \qquad \psi(x) = \sum_{i=1}^{n} x_i(1 \otimes e_i) : N^0 \to N^1.$$

*Write $q \perp 1 = q + X_0^2$ and let $y = x_0 = image(X_0) \in R(q \perp 1)$. Define*

$$\rho_M = \rho = \frac{1}{2}\begin{pmatrix} 1-y & \varphi(x) \\ -\psi(x) & 1+y \end{pmatrix} : N \to N.$$

*That means, for $n_0 \in N^0$, $n_1 \in N^1$ we have*

$$\rho(n_0, n_1) = \big((1-y)n_0 + \varphi(x)n_1, -\psi(x)n_0 + (1+y)n_1\big)/2.$$

*Then,*

$$\varphi\psi(x) = -q(x) : N^0 \to N^0, \qquad \psi\varphi(x) = -q(x) : N^1 \to N^1$$

*and $\rho$ is an idempotent homomorphism.*

**Proof.** By direct multiplication, it follows $\varphi\psi = -q(x)$, $\psi\varphi = -q(x)$. Again, we have

$$\rho^2 = \frac{1}{4}\begin{pmatrix} (1-y)^2 - \varphi(x)\psi(x) & 2\varphi(x) \\ -2\psi(x) & -\psi(x)\varphi(x) + (1+y)^2 \end{pmatrix}$$

which is

$$\frac{1}{4}\begin{pmatrix} (1-y)^2 + q(x) & 2\varphi(x) \\ -2\psi(x) & q(x) + (1+y)^2 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2(1-y) & 2\varphi(x) \\ -2\psi(x) & 2(1+y) \end{pmatrix} = \rho.$$

This completes the proof. □

**Definition 5.4.** We use the notation as in Proposition 5.3. Define a functor

$$\alpha : \mathcal{G}\big(C(-q)\big) \to \mathcal{P}\big(R(q \oplus 1)\big) \quad \text{by } \alpha(M) = kernel(\rho_M).$$

Since, $k \to R(q \perp 1)$ is flat, it follows easily that $\alpha$ is an exact functor. Therefore, $\alpha$ induces a homomorphism

$$\Theta_q : K\big(\mathcal{G}\big(C(-q)\big)\big) \to \widetilde{K_0}\big(R(q \perp 1)\big)$$

where $\forall M \in \mathcal{G}(C(-q))$

$$\Theta_q\big([M]\big) = \big[\alpha(M)\big] - rank\big(\alpha(M)\big).$$

Before we proceed, we will describe $\alpha(M)$ in (5.4) by patching two trivial bundles on the two (algebraic) hemispheres along the (algebraic) equator, as follows.

**Proposition 5.5.** *We will use all the notations of* (5.3, 5.4)*. We have* $q = q(X_1, \dots, X_n), q \perp 1 = q + X_0^2$ *and* $y = x_0 = image(X_0)$*. Let* $M = M^0 \oplus M^1 \in \mathcal{G}(C(-q))$, $N = N^0 \oplus N^1, \varphi, \psi$ *be as in* (5.3)*. Write*

$$F^0 = N^0_{1+y} = R\big(q + X_0^2\big)_{1+y} \otimes M^0, \qquad F^1 = N^1_{1-y} = R\big(q + X_0^2\big)_{1-y} \otimes M^1.$$

*Then* $\alpha(M)$ *is obtained by patching* $F^0$ *and* $F^1$ *via* $\psi_{1-y^2}$*. In particular, if* $k$ *is a field,* $\mathrm{rank}(\alpha(M)) = \dim_k M/2 = \dim_k M_0$.

**Proof.** Define

$$\sigma : F^0_{1-y} \to F^1_{1+y} \quad \text{by } \sigma(n_0) = \frac{-\psi(n_0)}{1+y}$$

and

$$\eta : F^1_{1+y} \to F^0_{1-y} \quad \text{by } \sigma(n_1) = \frac{\varphi(n_1)}{1-y}.$$

Then for $n_0 \in F^0_{1-y}$, we have

$$\eta\sigma(n_0) = \frac{-\varphi\psi(n_0)}{1-y^2} = \frac{q(x_1, \dots, x_n)(n_0)}{1-y^2} = n_0.$$

So, $\eta\sigma = 1$ and similarly, $\sigma\eta = 1$. Consider fiber product

$$\begin{array}{ccc} R(q \perp 1) & \longrightarrow & R(q \perp 1)_{1-y} \\ \downarrow & & \downarrow \\ R(q \perp 1)_{1+y} & \longrightarrow & R(q \perp 1)_{1-y^2} \end{array}$$

and define $P(\sigma)$ by the patching diagram

$$\begin{array}{ccccc} P(\sigma) & & \longrightarrow & & F^1 \\ \downarrow & & & & \downarrow \\ F^0 & \longrightarrow & F^0_{1-y} & \xrightarrow{\sigma} & F^1_{1+y}. \end{array}$$

Define

$$f_0 : F^0 = N^0_{1+y} \to \alpha(M)_{1+y} \quad \text{by } f_0(n_0) = \left(n_0, \frac{\psi(n_0)}{1+y}\right)$$

and

$$f_1 : F^1 = N^1_{1-y} \to \alpha(M)_{1-y} \quad \text{by } f_1(n_1) = -\left(\frac{-\varphi(n_1)}{1-y}, n_1\right).$$

We check that $f_0, f_1$ are well-defined isomorphisms. Recall that

$$\rho_M = \rho = \frac{1}{2}\begin{pmatrix} 1-y & \varphi(x) \\ -\psi(x) & 1+y \end{pmatrix}.$$

Using the identities $q(x) + y^2 = 1$, $\varphi\psi = -q$, $\psi\varphi = -q$, direct computation shows

$$f_0(n_0) = \left(n_0, \frac{\psi(n_0)}{1+y}\right), \qquad f_1(n_1) = -\left(\frac{-\varphi(n_1)}{1-y}, n_1\right) \in kernel(\rho) = \alpha(M).$$

So, $f_0$, $f_1$ are well defined. Clearly, $f_0$, $f_1$ are injective and their surjectivity can also be checked directly. Now consider the patching diagram:

$$\begin{tikzcd}
P(\sigma) \arrow[rrr] \arrow[dr, dashed, "f"] \arrow[dd] & & & F^1 \arrow[dr, "f_1"] \arrow[dd] & \\
& \alpha(M) \arrow[rrr] \arrow[dd] & & & \alpha(M)_{1-y} \arrow[dd] \\
F^0 \arrow[rr] \arrow[dr, "f_0"] & & F^0_{1-y} \arrow[r, "\sigma"] \arrow[dr, "f_0"] & F^1_{1+y} \arrow[dr, "f_1"] & \\
& \alpha(M)_{1+y} \arrow[rr] & & \alpha(M)_{1-y^2} \arrow[r, "Id"] & \alpha(M)_{1-y^2}.
\end{tikzcd}$$

We check $f_1\sigma = f_0$. For $n_0 \in F^0_{1-y}$, we have

$$\begin{aligned} f_1\sigma(n_0) &= -\left(\frac{-\varphi(\frac{-\psi(n_0)}{1+y})}{1-y}, \frac{-\psi(n_0)}{1+y}\right) \\ &= -\left(\frac{-q(x)}{1-y^2}(n_0), \frac{-\psi(n_0)}{1+y}\right) = \left(n_0, \frac{\psi(n_0)}{1+y}\right) = f_0(n_0), \end{aligned}$$

since $q(x) = 1 - y^2$. In this patching diagram above, $f$ is obtained by properties of fiber product diagrams. Now, since $f_0, f_1$ are isomorphisms, $f : P(\sigma) \to \alpha(M)$ is also an isomorphism. Let $P(\psi_{1-y^2})$ denote the projective module obtained by patching $F^0$ and $F^1$ via $\psi_{1-y^2}$. Since $P(\sigma) \approx P(\psi_{1-y^2})$, the proposition is established. $\square$

*5.2. Further background on Bott periodicity*

For the benefit of the readership, in this subsection, we give some further background on Bott periodicity from [ABS,Sw1]. We establish that $\Theta_q$ defined in (5.4) is a surjective homomorphism, when $q = \sum_{i=1}^{n} X_i^2 \in \mathbb{R}[X_1, \dots, X_n]$. In this case, $R(q \perp 1) = A_n$. We have the following proposition.

**Proposition 5.6.** *(See [ABS].) We continue to use notations as in* (5.3, 5.4)*. The composition*

$$K(\mathcal{G}(C(-q \perp -1))) \longrightarrow K(\mathcal{G}(C(-q))) \xrightarrow{\Theta_q} \widetilde{K_0}(R(q \perp 1))$$

*is zero. Further, as in* [ABS,Sw1]*, define* $ABS(q)$ *by the exact sequence*

$$K(\mathcal{G}(C(-q \perp -1))) \longrightarrow K(\mathcal{G}(C(-q))) \longrightarrow ABS(q) \longrightarrow 0.$$

*So, there is a homomorphism* $\alpha_q : ABS(q) \to \widetilde{K_0}(R(q \perp 1))$ *such that the diagram*

$$\begin{tikzcd}
K(\mathcal{G}(C(-q \perp -1))) \arrow[r] & K(\mathcal{G}(C(-q))) \arrow[r] \arrow[dr, "\Theta_q"'] & ABS(q) \arrow[r] \arrow[d, "\alpha_q"] & 0 \\
& & \widetilde{K_0}(R(q \perp 1)) &
\end{tikzcd}$$

*commute.*

**Proof.** We reinterpret the proof of Swan [Sw1, 7.7], and sketch a direct proof. Let $e_1, \dots, e_n$ denote the canonical generators of $C(-q)$ and $f$ be the other generator of $C(-q \perp -1)$. Let $M = M^0 \oplus M^1 \in \mathcal{G}(C(-q - Z^2))$. Write $N = M \otimes R(q \perp X_0^2)$ and define $f^* : M \to M$ such that $f^*_{|M^0} = f_{|M^0}$, $f^*_{|M^1} = -f_{|M^1}$ and similarly define $e_i^*$.

With the notations as in (5.3, 5.4), we have $\rho = \frac{1-\gamma}{2}$, where

$$\gamma = \begin{pmatrix} y & -\varphi \\ \psi & -y \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & -y \end{pmatrix} + \sum_{i=1}^{n} x_i e_i^*.$$

So, $\alpha(M) = \{w \in N\colon\ w = \gamma(w)\}$. Also note $\gamma^2 = 1$. Define

$$L^0 = \ker(f^* - 1) = \big\{(m_0, fm_0)\colon\ m_0 \in M^0\big\} = \big\{(-fm_1, m_1)\colon\ m_1 \in M^1\big\}$$

and

$$L^1 = \ker(f^* + 1) = \big\{(fm_1, m_1)\colon\ m_1 \in M^1\big\} = \big\{(m_0, -fm_0)\colon\ m_0 \in M^0\big\}.$$

So, $M = L^0 \oplus L^1$ and $N = Q^0 \oplus Q^1$ with $Q^0 = L^0 \otimes R(q \perp X_0^2)$, $Q^1 = L^1 \otimes R(q \perp X_0^2)$. We check that $\mathit{diagonal}(1, -1)L^0 \subseteq L^1$ and $e_i^* L^0 \subseteq L^1$. So, $\gamma(Q^0) \subseteq Q^1$ and similarly, $\gamma(Q^1) \subseteq Q^0$.

We have $Q^0 \cap \alpha(M) \subseteq Q^0 \cap Q^1 = 0$, and for $n = n^0 + n^1$ with $n^i \in Q^i$, we have $n = (n^0 - \gamma(n^1)) + (n^1 + \gamma(n^1)) \in Q^0 + \alpha(M)$. So, $N = Q^0 \oplus \alpha(M)$ and $\alpha(M) \approx N/Q^0 = Q^1$ is free. The proof is complete. □

The following theorem relates topological and algebraic $K$-groups.

**Theorem 5.7.** *(See [ABS,F,Sw1].) Let* $q = X_1^2 + \cdots + X_n^2 \in \mathbb{R}[X]$*. Then the following is a commutative diagram of isomorphisms*:

$$\begin{tikzcd}
ABS(q) \arrow[d, "\alpha_q"', "\wr"] \arrow[dr, "\sim"] & \\
\widetilde{K_0}(A_n) \arrow[r, "\sim"] & \widetilde{KO}(\mathbb{S}^n).
\end{tikzcd}$$

*In particular, the homomorphism*

$$\Theta_q : K\big(\mathcal{G}\big(C(-q)\big)\big) \to \widetilde{K_0}\big(R(q \perp 1)\big) \text{ is surjective.}$$

**Proof.** Note that $R(q \perp 1) = \frac{\mathbb{R}[X_0, X_1, \ldots, X_n]}{(X_0^2 + q - 1)} = A_n$. The diagonal isomorphism is established in [ABS]. The horizontal (equivalently the vertical) homomorphism is an isomorphism due to the theorem of Swan [Sw2, Theorem 3]. So, the proof is complete. □

*5.3. Patching matrices*

Proposition 5.5 exhibits the importance of a suitable description of the homomorphism $\psi_{1-y^2}$, as a matrix. We are interested in the cases of real spheres $Spec(A_n)$ with $K_0(A_n) \approx KO(\mathbb{S}^n)$ nontrivial. So, $q = \sum_{i=1}^n X_i^2 \in \mathbb{R}[X_1, \ldots, X_n]$ and $n = 8r, 8r+1, 8r+2, 8r+4$. We only need to consider the irreducible $\mathbb{Z}_2$-graded modules $M$ over $C_n = C(-q)$.

We will include the following from [ABS] regarding Bott periodicity.

**Theorem 5.8.** *Let $q_n = q = \sum_{i=1}^n X_i^2 \in \mathbb{R}[X_1, \ldots, X_n]$ and $C_n = C(-q)$. Write $a_n = (\dim_{\mathbb{R}} \mathcal{I}_n)/2 = \dim_{\mathbb{R}} \mathcal{I}_n^0$ where $\mathcal{I}_n = \mathcal{I}_n^0 \oplus \mathcal{I}_n^1$ is an irreducible $\mathbb{Z}_2$-graded $C_n$-module. The following chart summarizes some information regarding $C_n = C(-q)$:*

| $n$ | $C_n$ | $K(\mathcal{G}(C_n))$ | $ABS(q_n)$ | $a_n$ |
|---|---|---|---|---|
| 1 | $\mathbb{C}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 1 |
| 2 | $\mathbb{H}$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | 2 |
| 3 | $\mathbb{H} \oplus \mathbb{H}$ | $\mathbb{Z}$ | 0 | 4 |
| 4 | $\mathbb{M}_2(\mathbb{H})$ | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | 4 |
| 5 | $\mathbb{M}_4(\mathbb{C})$ | $\mathbb{Z}$ | 0 | 8 |
| 6 | $\mathbb{M}_8(\mathbb{R})$ | $\mathbb{Z}$ | 0 | 8 |
| 7 | $\mathbb{M}_8(\mathbb{R}) \oplus \mathbb{M}_8(\mathbb{R})$ | $\mathbb{Z}$ | 0 | 8 |
| 8 | $\mathbb{M}_{16}(\mathbb{R})$ | $\mathbb{Z} \oplus \mathbb{Z}$ | $\mathbb{Z}$ | 8 |

*Further*

$$C_{n+8} \approx C_8 \otimes C_n \approx \mathbb{M}_{16}(\mathbb{R}) \otimes C_n$$

*and*

$$K\big(\mathcal{G}(C_{n+8})\big) \approx K\big(\mathcal{G}(C_n)\big), \qquad ABS(q_{n+8}) \approx ABS(q_8), \qquad a_{n+8} = 16a_n.$$

The following corollary will be of some interest to us.

**Corollary 5.9.** *With notations as above* (5.8)*, for nonnegative integers $r$, we have*

$$C_{8r} \approx \mathbb{M}_{16^r}(\mathbb{R}), \qquad C_{8r+1} \approx \mathbb{M}_{16^r}(\mathbb{C}), \qquad C_{8r+2} \approx \mathbb{M}_{16^r}(\mathbb{H}), \qquad C_{8r+4} \approx \mathbb{M}_{2*16^r}(\mathbb{H})$$

*and*

$$a_{8r} = 16^r/2, \qquad a_{8r+1} = 16^r, \qquad a_{8r+2} = 2 * 16^r, \qquad a_{8r+4} = 4 * 16^r.$$

**Proof.** First part follows from (5.8). For the later part, let $I_n$ be an irreducible $C_n$-module. Note that for $n = 8r, 8r+1, 8r+2, 8r+4$, Clifford algebras $C_n$ are matrix algebras. Form general theory, $I_n$ is isomorphic to the module of column vectors. So, $\dim_{\mathbb{R}} I_n$ is easily computable. One can also establish, by induction, that there are $\mathbb{Z}_2$-graded $C_n$-modules $\mathcal{I}_n$ with $\dim \mathcal{I}_n = \dim I_n$. So, $\mathcal{I}_n$ is irreducible and $a_n = (\dim \mathcal{I}_n)/2 = (\dim I)/2$. This completes the proof. □

**Theorem 5.10.** *Following chart describes the $\widetilde{KO}(\mathbb{S}^n)$ groups.*

| $n$ | $8r$ | $8r+1$ | $8r+2$ | $8r+3$ | $8r+4$ | $8r+5$ | $8r+6$ | $8r+7$ |
|---|---|---|---|---|---|---|---|---|
| $\widetilde{KO}(\mathbb{S}^n)$ | $\mathbb{Z}$ | $\mathbb{Z}_2$ | $\mathbb{Z}_2$ | 0 | $\mathbb{Z}$ | 0 | 0 | 0 |

**Proof.** It follows from Theorem 5.7 and Corollary 5.9. For a complete proof the reader is referred to the book [H] or [ABS]. $\square$

Now we state our result on the matrix representation of $\psi$. In fact, we will do it more formally at the polynomial ring level.

**Proposition 5.11.** *Let $n = 8r, 8r+1, 8r+2, 8r+4$ be a nonnegative integer and $q(X) = \sum_{i=1}^n X_i^2 \in \mathbb{R}[X_1,\dots,X_n]$. As before, $C_n = C(-q)$ and $e_1,\dots,e_n$ are the canonical generators of $C_n$.*

*Let $M = M^0 \oplus M^1$ be a $\mathbb{Z}_2$-graded irreducible $C_n$-module. Write $m = a_n = \dim_{\mathbb{R}} M^0$. Define*

$$\Psi = \Psi_n = \sum_{i=1}^n X_i(1 \otimes e_i) : \mathbb{R}[X_1,\dots,X_n] \otimes M^0 \to \mathbb{R}[X_1,\dots,X_n] \otimes M^1$$

*and*

$$\Phi = \Phi_n = \sum_{i=1}^n X_i(1 \otimes e_i) : \mathbb{R}[X_1,\dots,X_n] \otimes M^1 \to \mathbb{R}[X_1,\dots,X_n] \otimes M^0.$$

*Then, there are choices of bases $u_1,\dots,u_m$ of $M^0$ and $v_1,\dots,v_m$ of $M^1$ such that the matrix $\Gamma$ of $\Psi$ and the matrix $\Delta$ of $\Phi$ have the following properties:*

1. *Each row and column of $\Gamma$, $\Delta$ has exactly $n$ nonzero entries and for $i = 1,\dots,n$ exactly one entry in each row and column is $\pm X_i$.*
2. *As a consequence, $\Delta = -\Gamma^t$ and they are orthogonal matrices.*

**Proof of (1) $\Rightarrow$ (2).** Suppose we have bases of $M^0, M^1$ as above that satisfy (1). Write $u = (u_1,\dots,u_m)^t$, $v = (v_1,\dots,v_m)^t$. Then we have $-q(x)(u) = \Phi(u) = \Gamma\Delta(v)$. So, $\Gamma\Delta = -q$. Let $\Gamma_i^r$ denote the $i$th-row of $\Gamma$ and $\Delta_i^c$ denote the $i$th-column of $\Delta$. So, $\Gamma_i^r\Delta_i^c = -\sum_{i=1}^n X_i^2$. Comparing two sides, we have $\Gamma_i^r = -(\Delta_i^c)^t$. So, $\Delta = -\Gamma^t$. Since $\Gamma\Delta = -q$, we have $\Gamma, \Delta$ are orthogonal matrices. Proof of (1) comes later. $\square$

Before we get into the proof of 5.11, we wish to deal with the initial cases of $n = 2, 4, 8$ with some extra details.

**Lemma 5.12.** *Let $q = X_1^2 + X_2^2 \in \mathbb{R}$ and $C_2 = C(-q)$. Then $C_2 = \mathbb{H}$, where the canonical basis of $\mathbb{R}^2 \subseteq C_2$ is $e_1 = i$, $e_2 = j$ and the matrices of $\Psi_2$ and $\Phi_2$ have the property* (1) *of Proposition* 5.11.

**Proof.** A matrix representation of $\Psi$ is given by

$$\begin{pmatrix} \Psi(1) \\ \Psi(k) \end{pmatrix} = \begin{pmatrix} X_1 & X_2 \\ X_2 & -X_1 \end{pmatrix} \begin{pmatrix} i \\ j \end{pmatrix}.$$

Similarly, we can get a matrix representation of $\Phi$. The proof is complete. $\square$

Now we consider the case $n = 4$. We will include additional information that will be useful later.

**Lemma 5.13.** *Let* $q = \sum_{i=1}^{4} X_i^2 \in \mathbb{R}[X_1, X_2, X_3, X_4]$ *and* $C_4 = C(-q)$. *Then*:

1. *We have* $C_4 = \mathbb{M}_2(\mathbb{H})$ *where the canonical basis of* $\mathbb{R}^4 \subseteq C_4$ *is given as follows*:

$$e_1 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \qquad e_2 = \begin{pmatrix} j & 0 \\ 0 & j \end{pmatrix},$$

*and*

$$e_3 = \begin{pmatrix} 0 & k \\ k & 0 \end{pmatrix}, \qquad e_4 = \begin{pmatrix} k & 0 \\ 0 & -k \end{pmatrix}.$$

2. *Following* [F], *write* $w_4 = 1 + e_1e_2e_3e_4$. *We have the following identities,*

$$e_1e_2e_3e_4w_4 = w_4, \qquad -e_1e_2w_4 = e_3e_4w_4, \qquad e_1e_3w_4 = e_2e_4w_4, \qquad -e_1e_4w_4 = e_2e_3w_4.$$

3. *Let* $M = C_4w_4$. *Then,* $M$ *is irreducible.*
4. *Then* $\Psi_4$, $\Phi_4$ *have the desired property* (1) *of* (5.11).

**Proof.** Proof of (1) follows by direct checking. Identities in (2) are obvious. The statement (3) is a theorem of Fossum [F]. To see a proof, let $M = Cw_4 = M^0 \oplus M^1$ be the $\mathbb{Z}_2$-graded decomposition of $M$. We have

$$M^0 = \mathbb{R}w_4 + \sum_{i<j} \mathbb{R}e_ie_jw_4 + \mathbb{R}e_1e_2e_3e_4w_4; \qquad M^1 = \sum \mathbb{R}e_iw_4 + \sum_{i<j<k} \mathbb{R}e_ie_je_kw_4.$$

Using the identities above, it is easy to check that a basis of $M^0$ is given by

$$u_1 = w_4, \qquad u_2 = -e_1e_2w_4, \qquad u_3 = -e_1e_3w_4, \qquad u_4 = -e_1e_4w_4$$

and a basis of $M^1$ is given by

$$v_1 = e_1w_4 = e_1u_1, \qquad v_2 = e_2w_4, \qquad v_3 = e_3w_4, \qquad v_4 = e_4w_4 = e_1e_2e_3w_4.$$

Since dimension of an irreducible module over $C_4 = \mathbb{M}_2(\mathbb{H})$ is eight, $M$ is irreducible. This establishes (3).

Now, we write down the matrix of $\Psi_4, \Phi_4$ with respect to the above bases:

$$\begin{pmatrix} \Psi(u_1) \\ \Psi(u_2) \\ \Psi(u_3) \\ \Psi(u_4) \end{pmatrix} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \\ -X_2 & X_1 & X_4 & -X_3 \\ -X_3 & -X_4 & X_1 & X_2 \\ -X_4 & X_3 & -X_2 & X_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}.$$

Also

$$\begin{pmatrix} \Phi(v_1) \\ \Phi(v_2) \\ \Phi(v_3) \\ \Phi(v_4) \end{pmatrix} = \begin{pmatrix} -X_1 & X_2 & X_3 & X_4 \\ -X_2 & -X_1 & X_4 & -X_3 \\ -X_3 & -X_4 & -X_1 & X_2 \\ -X_4 & X_3 & -X_2 & -X_1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}.$$

The proof is complete. □

Now we will consider the case of $n = 8$.

**Lemma 5.14.** *Let $q = \sum_{i=1}^{8} X_i^2 \in \mathbb{R}[X_1, \ldots, X_8]$ and $C_8 = C(-q)$. Let $E_1, E_2, \ldots, E_8$ be the canonical generators of $C_8$. Following Fossum* [F], *let*

$$w_8 = (1 + E_1E_2E_5E_6 + E_1E_3E_5E_7 + E_1E_4E_5E_8)(1 + E_1E_2E_3E_4)(1 + E_5E_6E_7E_8).$$

*Then $M = C_8w_8$ is irreducible and $\Psi_8$, $\Phi_8$ have the desired property* (1) *of* (5.11)*.*

**Proof.** It is a theorem of Fossum [F], that $M = C_8w_8$ is irreducible. A basis of $M$ is also given in [F]. We will describe this basis of $M$ and provide a proof of irreducibility. By (5.1), there is an isomorphism $C_8 \approx C_4 \widehat{\otimes} C_4$. As in (5.13), we denote the canonical generators of $C_4$ by $e_1$, $e_2$, $e_3$, $e_4$ and $w_4 = 1 + e_1e_2e_3e_4$.

We will identify $C_8 = C_4 \widehat{\otimes} C_4$. Under this identification, the canonical generators of $C_8$ are given by $E_i = e_i \otimes 1$ for $i = 1, 2, 3, 4$ and $E_i = 1 \otimes e_{i-4}$ for $i = 5, 6, 7, 8$. Also, $w_8$ is identified as

$$w_8 = (w_4 \otimes w_4 + e_1e_2w_4 \otimes e_1e_2w_4 + e_1e_3w_4 \otimes e_1e_3w_4 + e_1e_4w_4 \otimes e_1e_4w_4).$$

We denote $w = w_8$. Let $M = C_8w_8 = M^0 \oplus M^1$ be the $\mathbb{Z}_2$-graded decomposition of $M$. We denote the basis [F] of $M^0$ by $u_i$ as in the table:

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | $u_8$ |
|---|---|---|---|---|---|---|---|---|
| $=$ | $w$ | $E_1E_2w$ | $E_1E_3w$ | $E_2E_3w$ | $E_1E_5w$ | $E_2E_5w$ | $E_3E_5w$ | $E_1E_2E_3E_5w$ |

and similarly, $v_i$ will denote the basis [F] of $M^1$ as follows:

| | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ |
|---|---|---|---|---|---|---|---|---|
| $=$ | $E_1w$ | $E_2w$ | $E_3w$ | $E_1E_2E_3w$ | $E_5w$ | $E_1E_2E_5w$ | $E_1E_3E_5w$ | $E_2E_3E_5w$ |

The following multiplication table will be useful for our purpose:

| | $u_1$ | $u_2$ | $u_3$ | $u_4$ | $u_5$ | $u_6$ | $u_7$ | $u_8$ |
|---|---|---|---|---|---|---|---|---|
| $E_1$ | $v_1$ | $-v_2$ | $-v_3$ | $v_4$ | $-v_5$ | $v_6$ | $v_7$ | $-v_8$ |
| $E_2$ | $v_2$ | $v_1$ | $-v_4$ | $-v_3$ | $-v_6$ | $-v_5$ | $v_8$ | $v_7$ |
| $E_3$ | $v_3$ | $v_4$ | $v_1$ | $v_2$ | $-v_7$ | $-v_8$ | $-v_5$ | $-v_6$ |
| $E_4$ | $v_4$ | $-v_3$ | $v_2$ | $-v_1$ | $v_8$ | $-v_7$ | $v_6$ | $-v_5$ |
| $E_5$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ |
| $E_6$ | $v_6$ | $-v_5$ | $-v_8$ | $v_7$ | $v_2$ | $-v_1$ | $-v_4$ | $v_3$ |
| $E_7$ | $v_7$ | $v_8$ | $-v_5$ | $-v_6$ | $v_3$ | $v_4$ | $-v_1$ | $-v_2$ |
| $E_8$ | $-v_8$ | $v_7$ | $-v_6$ | $v_5$ | $v_4$ | $-v_3$ | $v_2$ | $-v_1$ |

This table is constructed by using the identities in (5.13). For the benefit of the reader, we give proof of one of them. We prove $E_6u_1 = E_6w = v_6$. First, we have $E_6w = -E_5(E_5E_6w) = -E_5(1 \otimes e_1e_2)w$. We compute

$$\begin{aligned} E_5E_6w &= (1 \otimes e_1e_2)w \\ &= (w_4 \otimes e_1e_2w_4 + e_1e_2w_4 \otimes e_1e_2e_1e_2w_4 + e_1e_3w_4 \otimes e_1e_2e_1e_3w_4 + e_1e_4w_4 \otimes e_1e_2e_1e_4w_4) \\ &= (-e_1e_2 \otimes 1)[e_1e_2w_4 \otimes e_1e_2w_4 + w_4 \otimes w_4 + e_2e_3w_4 \otimes e_2e_3w_4 + e_2e_4w_4 \otimes e_2e_4w_4] \\ &= -E_1E_2\big[e_1e_2w_4 \otimes e_1e_2w_4 + w_4 \otimes w_4 + (-e_1e_4w_w \otimes -e_1e_4w_4) + e_1e_3w_4 \otimes e_1e_3w_4\big]. \end{aligned}$$

Therefore, $E_5E_6w = (1 \otimes e_1e_2)w = -E_1E_2w$. So,

$$E_6w = -E_5(E_5E_6w) = -E_5(-E_1E_2w) = E_1E_2E_5w = v_6.$$

This establishes $E_6u_1 = v_6$.

A similar multiplication table $(E_iv_j)$ can be constructed using the fact $E_iv_j = u_k \Leftrightarrow -v_j = E_iu_k$. This shows that the vector space $V$ generated by $\{u_i, v_j\colon\ i, j = 1, \ldots, 8\}$ is a $C_8$-(left)module. So $V = M$. Since, an irreducible $C_8$-module has dimension sixteen, $M$ is irreducible.

Now we compute the matrix $\Gamma$ of $\Psi$ with respect to these bases. We have, $\Psi(u_i) = \sum_{j=1}^{8} X_jE_ju_i$. The $(i, j)$th entry of the matrix $\Gamma$ of $\Psi$ is $\pm X_k$ if and only if $E_ku_i = \pm v_j$. For a fixed $i$ there is exactly one $j$ such that $E_ku_i = \pm v_j$ and similarly for a fixed $j$ there is exactly one $i$ such that $E_ku_i = \pm v_j$. So, for $k = 1, \ldots, 8$; $\pm X_k$ appears exactly once in each row and column. In fact, The matrix of $\Psi$ is

$$\Gamma = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 & X_5 & X_6 & X_7 & -X_8 \\ X_2 & -X_1 & -X_4 & X_3 & -X_6 & X_5 & X_8 & X_7 \\ X_3 & X_4 & -X_1 & -X_2 & -X_7 & -X_8 & X_5 & -X_6 \\ -X_4 & X_3 & -X_2 & X_1 & X_8 & -X_7 & X_6 & X_5 \\ X_5 & X_6 & X_7 & X_8 & -X_1 & -X_2 & -X_3 & X_4 \\ -X_6 & X_5 & -X_8 & X_7 & -X_2 & X_1 & -X_4 & -X_3 \\ -X_7 & X_8 & X_5 & -X_6 & -X_3 & X_4 & X_1 & X_2 \\ -X_8 & -X_7 & X_6 & X_5 & -X_4 & -X_3 & X_2 & -X_1 \end{pmatrix}.$$

Similar argument can be given for $\Phi$. So, $\Psi$, $\Phi$ have the property (1) of (5.11).

We remark that the property (2) of (5.11) was established as a consequence of property (1). Alternately, we can use the fact $E_iv_j = u_k \Leftrightarrow -v_j = E_iu_k$ to establish property (2). This completes the proof of (5.14). $\square$

Now we are ready to give a complete proof of Proposition 5.11.

**Proof of 5.11.** We already proved (1) $\Rightarrow$ (2). So, we only need to prove (1). The case $n = 1$, is obvious and Lemmas 5.12, 5.13, 5.14, respectively, establish the proposition in the cases $n = 2, 4, 8$.

We will use induction, i.e. we assume that (1) of the proposition is valid for some $m = 8r$, $8r + 1, 8r + 2, 8r + 4$ and prove that the same is valid for $n = m + 8$.

First, we set up some notations. For a matrix $A$, the $i$th-row will be denoted by ${}_rA_i$ and the $i$th-column will be denoted by ${}_cA_i$. We have $m + 8$ variables $X_1, \ldots, X_m, X_{m+1}, \ldots, X_{m+8}$. For $i = 1, \ldots, 8$, we will write $Y_i = X_{m+i}$. In our cases, there is only one irreducible $\mathbb{Z}_2$-graded $C_m$-module, which will be denoted by $M(m) = M(m)^0 \oplus M(m)^1$. We have $C_{m+8} = C_m \widehat{\otimes} C_8$. Comparing dimensions (see 5.9), we have $M(m + 8) = M(m) \widehat{\otimes} M(8)$.

Write $N = \dim_{\mathbb{R}} M(m)^0$. We assume that there are bases $u_1, \ldots, u_N$ of $M(m)^0$ and $v_1, \ldots, v_N$ of $M(m)^1$ and bases $\mu_1, \ldots, \mu_8$ of $M(8)^0$ and $\nu_1, \ldots, \nu_8$ of $M(8)^1$ such that

$$\begin{pmatrix} \Psi_m(u_1) \\ \cdots \\ \cdots \\ \Psi_m(u_N) \end{pmatrix} = A(X) \begin{pmatrix} v_1 \\ \cdots \\ \cdots \\ v_N \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Psi_8(\mu_1) \\ \cdots \\ \cdots \\ \Psi_8(\mu_8) \end{pmatrix} = B(Y) \begin{pmatrix} \nu_1 \\ \cdots \\ \cdots \\ \nu_8 \end{pmatrix}$$

where $A(X) = (a_{ij}(X_1, \ldots, X_m))$ and $B(Y) = (b_{ij}(Y_1, \ldots, Y_8))$ have the properties $\Gamma$ of the proposition. Also

$$M(m + 8)^0 = M(m)^0 \otimes M(8)^0 \oplus M(m)^1 \otimes M(8)^1 \quad \text{with basis } u_i \otimes \mu_j,\ v_i \otimes \nu_j$$

and

$$M(m + 8)^1 = M(m)^1 \otimes M(8)^0 \oplus M(m)^0 \otimes M(8)^1 \quad \text{with basis } v_i \otimes \mu_j,\ u_i \otimes \nu_j.$$

The canonical generators of $C_m$ will be denoted by $e_1, \dots, e_m$ and the canonical generators of $C_8$ will be denoted by $e'_1, \dots, e'_8$. With $E_1 = e_1 \otimes 1, \dots, E_m = e_m \otimes 1$; $E'_1 = E_{m+1} = 1 \otimes e'_1, \dots, E'_8 = E_{m+8} = 1 \otimes e'_8$, we have

$$\Psi_{m+8} = \sum_{i=1}^{N} X_i E_i + \sum_{i=1}^{8} Y_i E'_i.$$

We have

$$\begin{aligned}\Psi_{m+8}(u_1 \otimes \mu_1) &= \sum_{i=1}^{N} X_i E_i (u_1 \otimes \mu_1) + \sum_{i=1}^{8} Y_i E'_i (u_1 \otimes \mu_1) \\ &= \sum_{i=1}^{N} a_{1i}(X) v_i \otimes \mu_1 + \sum_{i=1}^{8} b_{1i}(Y) u_1 \otimes \nu_i.\end{aligned}$$

We will use the notations $u = (u_1, \dots, u_N)^t$, $v = (v_1, \dots, v_N)^t$, $\mu = (\mu_1, \dots, \mu_8)^t$, $\nu = (\nu_1, \dots, \nu_8)^t$. With these notation,

$$\Psi_{m+8}(u_1 \otimes \mu_1) = {}_rA(X)_1 v \otimes \mu_1 + {}_rB(Y)_1 u_1 \otimes \nu.$$

For $i = 1, \dots, N$, likewise, we get

$$\Psi_{m+8}(u_i \otimes \mu_1) = {}_rA(X)_i v \otimes \mu_1 + {}_rB(Y)_1 u_i \otimes \nu.$$

Therefore,

$$\Psi_{m+8}(u \otimes \mu_1) = \left(\begin{array}{cccc|cccc} {}_rA_1(X) & 0 & \dots & 0 & {}_rB_1(Y) & 0 & 0 & 0 \\ {}_rA_2(X) & 0 & \dots & 0 & 0 & {}_rB_1(Y) & 0 & 0 \\ \dots & 0 & \dots & 0 & \dots & \dots & \dots & \\ {}_rA_N(X) & 0 & \dots & 0 & 0 & 0 & 0 & {}_rB_1(Y) \end{array}\right) \begin{pmatrix} v \otimes \mu_1 \\ \dots \\ v \otimes \mu_8 \\ u_1 \otimes \nu \\ \dots \\ u_N \otimes \nu \end{pmatrix}.$$

Given a row vector $a(Y)$ of length 8, let $\mathcal{R}(a(Y)) \in \mathbb{M}_{N \times 8N}$ denotes the matrix as on the right-hand side of the above matrix. With such notations,

$$\Psi_{m+8}(u \otimes \mu_1) = \left(\begin{array}{cccc|c} A(X) & 0 & \dots & 0 & \mathcal{R}\big({}_rB_1(Y)\big) \end{array}\right) \begin{pmatrix} v \otimes \mu_1 \\ \dots \\ v \otimes \mu_8 \\ u_1 \otimes \nu \\ \dots \\ u_N \otimes \nu \end{pmatrix}.$$

Using similar calculations for $\Psi_{m+8}(u \otimes \mu_i)$ we have

$$\begin{pmatrix} \Psi_{m+8}(u \otimes \mu_1) \\ \Psi_{m+8}(u \otimes \mu_2) \\ \dots \\ \Psi_{m+8}(u \otimes \mu_8) \end{pmatrix} = \left(\begin{array}{cccc|c} A(X) & 0 & \dots & 0 & \mathcal{R}({}_rB_1(Y)) \\ 0 & A(X) & \dots & 0 & \mathcal{R}({}_rB_2(Y)) \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & A(X) & \mathcal{R}({}_rB_8(Y)) \end{array}\right) \begin{pmatrix} v \otimes \mu_1 \\ \dots \\ v \otimes \mu_8 \\ u_1 \otimes \nu \\ \dots \\ u_N \otimes \nu \end{pmatrix}.$$

This gives the upper $8N$ rows of the matrix of $\Psi_{m+8}$ which form a $8N \times (8N + 8N)$ matrix. Now we proceed to compute

$$\begin{pmatrix} \Psi_{m+8}(v_1 \otimes v) \\ \Psi_{m+8}(v_2 \otimes v) \\ \cdots \\ \Psi_{m+8}(v_N \otimes v) \end{pmatrix}.$$

Again, the matrices of $\Phi_m$, $\Phi_8$ are respectively $-A(X)^t, -B(Y)^t$. We have,

$$\begin{aligned} \Psi_{m+8}(v_1 \otimes v_1) &= \sum_{i=1}^{m} X_i(e_i \otimes 1)(v_1 \otimes v_1) + \sum_{i=1}^{8} Y_i\bigl(1 \otimes e_i'\bigr)(v_1 \otimes v_1) \\ &= \Phi_m(v_1) \otimes v_1 - v_1 \otimes \Phi_8(v_1) = -\sum_{j=1}^{N} a_{j1}(X)u_j \otimes v_1 + \sum_{j=1}^{8} b_{j1}(Y)v_1 \otimes \mu_j \\ &= -{}_rA_1(X)^t u \otimes v_1 + {}_rB_1(Y)^t v_1 \otimes \mu. \end{aligned}$$

Similarly, for $i = 1, \dots, 8$, we have

$$\Psi_{m+8}(v_1 \otimes v_i) = \Phi_m(v_1) \otimes v_i - v_1 \otimes \Phi_8(v_i) = -{}_rA_1(X)^t u \otimes v_i + {}_rB_i(Y)^t v_1 \otimes \mu;$$

and for $k = 1, \dots, N$, we have

$$\Psi_{m+8}(v_k \otimes v_i) = \Phi_m(v_k) \otimes v_i - v_k \otimes \Phi_8(v_i) = -{}_rA_k(X)^t u \otimes v_i + {}_rB_i(Y)^t v_k \otimes \mu.$$

So, the left half of the matrix of $\Psi_{m+8}(v_1 \otimes v)$ is given by

$$\left( {}_cB_1^t \quad 0 \quad \cdots \quad 0 \;\middle|\; {}_cB_2^t \quad 0 \quad \cdots \quad 0 \;\middle|\; \cdots \;\middle|\; {}_cB_8^t \quad 0 \quad \cdots \quad 0 \right) \in \mathbb{M}_{8\times 8N}.$$

Similarly, the left half of the matrix of $\Psi_{m+8}(v_2 \otimes v)$ is given by

$$\left( 0 \quad {}_cB_1^t \quad \cdots \quad 0 \;\middle|\; 0 \quad {}_cB_2^t \quad \cdots \quad 0 \;\middle|\; \cdots \;\middle|\; 0 \quad {}_cB_8^t \quad \cdots \quad 0 \right) \in \mathbb{M}_{8\times 8N}.$$

So, the left-lower block of the matrix $\Psi_{m+8}$ is given by

$$\left(\begin{array}{cccc|c|cccc} {}_cB_1(Y)^t & 0 & \cdots & 0 & \cdots & {}_cB_8(Y)^t & 0 & \cdots & 0 \\ 0 & {}_cB_1(Y)^t & \cdots & 0 & \cdots & 0 & {}_cB_8(Y)^t & \cdots & 0 \\ 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & {}_cB_1(Y)^t & \cdots & 0 & 0 & \cdots & {}_cB_8(Y)^t \end{array}\right)$$

$$= \begin{pmatrix} \mathcal{R}({}_rB_1(Y)) \\ \mathcal{R}({}_rB_2(Y)) \\ \cdots \\ \mathcal{R}({}_rB_8(Y)) \end{pmatrix}^t = (\textit{Upper-Right Block})^t \in \mathbb{M}_{8N\times 8N}.$$

Now we compute the right half of the same matrix of

$$\begin{pmatrix} \Psi_{m+8}(v_1 \otimes v) \\ \Psi_{m+8}(v_2 \otimes v) \\ \cdots \\ \Psi_{m+8}(v_N \otimes v) \end{pmatrix}.$$

Let $\mathbb{I}_8$ denote the identity matrix of order 8. Then, the right half of this matrix is given by

$$-\begin{pmatrix} a_{11}(X)\mathbb{I}_8 & a_{21}(X)\mathbb{I}_8 & \cdots & a_{N1}(X)\mathbb{I}_8 \\ a_{12}(X)\mathbb{I}_8 & a_{22}(X)\mathbb{I}_8 & \cdots & a_{N2}(X)\mathbb{I}_8 \\ \cdots & \cdots & \cdots & \cdots \\ a_{1N}(X)\mathbb{I}_8 & a_{2N}(X)\mathbb{I}_8 & \cdots & a_{NN}(X)\mathbb{I}_8 \end{pmatrix}.$$

Now, the upper left and lower right blocks of the matrix of $\Psi_{m+8}$ involve only the variables $X_1,\dots,X_m$ and the upper right and lower left blocks involve only the variables $Y_1,\dots,Y_8$. Recall that $A(X)$ and $B(Y)$ have the properties of $\Gamma$ of the proposition. Examining all four blocks of the matrix of $\Psi_{m+8}$, it follows that the matrix of $\Psi_{m+8}$ also has the property of $\Gamma$ of the proposition. By symmetry, the matrix of $\Phi_{m+8}$ also has the property of $\Delta$ of the proposition. This completes the proof of Proposition 5.11. $\square$

## 6. Complete intersections

In this final section, we consider the question whether a local complete intersection ideal $I$ of $A_n$, with $height(I) = n$, is the image of a projective $A_n$-module $P$ of rank $n$. For an affirmative answer to this question for all such ideals $I$ it is necessary that the top Chern class map $C_0 : K_0(A_n) \to CH_0(A_n)$ is surjective. Since $CH_0(A_n) = \mathbb{Z}_2$ and for $n = 8r+3, 8r+5, 8r+6, 8r+7$, by (5.10), $K_0(A_n) = \mathbb{Z}$, the top Chern class map $C_0$ fails to be surjective. So, in these cases, the question has a negative answer.

We consider the stronger question whether each element of the Euler class group $E(A_n) = \mathbb{Z}$ is Euler class of a projective $A_n$-module $P$ of rank $n$. We start with the following two theorems about even classes and odd classes.

**Theorem 6.1.** *Let $A_n$ be the ring of algebraic functions on $\mathbb{S}^n$, as in nation* (2.1)*, and $n \geqslant 2$ be* ***even****. Let $N = 2r$ be an even integer. Then there is a stably free $A_n$-module $P$ of rank $n$ and an orientation $\chi : A_n \xrightarrow{\sim} \det(P)$ such that $e(p, \chi) = N$.*

**Proof.** By Lemma 2.4, we can assume $N \geqslant 0$. Let $m_1,\dots,m_N$ be even number of distinct real maximal ideals and assume that the corresponding real points are on $\mathbb{S}^1 = (x_2 = 0,\dots,x_n = 0)$. As in Lemma 4.1,

$$\bigcap_{i=1}^{N} m_i = (L, x_2, \dots, x_n) \quad \text{where } L = \prod_{i=1}^{N/2} L_i \quad \text{with } L_i \text{ linear.}$$

Write $F = A_n^n$ and $J = \bigcap_{i=1}^{N} m_i = (L, x_2, \dots, x_n)$. The standard basis of $F$ will be denoted by $e_1,\dots,e_n$. Define

$$f : F \twoheadrightarrow J \quad \text{by } f(e_1) = L, \quad f(e_i) = x_i \quad \forall i \geqslant 2.$$

Let

$$\omega : F/JF \to J/J^2 \quad \text{and} \quad \text{for } i = 0, 1 \quad \omega_i : F/m_iF \to m_i/m_i^2,$$

be induced by $f$. Therefore

$$(J, \omega) = \sum_{i=1}^{N} (m_i, \omega_i) = 0 \in E(A_n).$$

We can assume

$$(m_i,\omega_i)=1\quad\forall i=1,\dots,s,\qquad (m_i,\omega_i)=-1\quad\forall i=s+1,\dots,N.$$

Let $u\in A$ be such that $u-1\in m_i$ for $i=1,\dots,s$ and $u+1\in m_i$ for $i=s+1,\dots,N$. Note $u^2-1\in J$. Define $P$ by the exact sequence

$$0\to P\to A_n\oplus F\xrightarrow{(u,-f)}A_n\to 0.$$

By [BRS2, Lemma 5.1], $P$ has an orientation $\chi$ such that

$$e(P,\chi)=u^{-(n-1)}(J,\omega)=\sum_{i=1}^{s}(m_i,\omega_i)+\sum_{i=s+1}^{N}-(m_i,\omega_i)=N.$$

The proof is complete. □

**Theorem 6.2.** *Let $A_n$ be the ring of algebraic functions on $\mathbb{S}^n$. Assume $n=\dim A_n$ is* ***even****. Then, there exists a projective $A_n$-module $P$ with $\mathrm{rank}(P)=n$ and an orientation $\chi:A_n\xrightarrow{\sim}\det P$ with $e(P,\chi)=N$ for some odd integer $N$ if and only if the same is possible for all odd integers $N$.*

**Proof.** Suppose $N$ odd and $e(P,\chi)=N$. By Lemma 2.4, we can assume $N>0$. Now assume $M$ be any other odd integer. Again, we can assume $M>0$. Let $m_1,\dots,m_M$ be distinct real maximal ideals and $(m_i,\omega_i)=1\in EL(A_n)=\mathbb{Z}$. Write $F=A_n^n$ and $I=\bigcap_{i=1}^M m_i$. Let $\omega_I:F/IF\to I/I^2$ be obtained from $\omega_1,\dots,\omega_M$. Then $M=(I,\omega_I)$. Note that the weak Euler class group

$$E_0(A_n)\approx CH_0(A_n)=\mathbb{Z}/(2).$$

Therefore, the weak Euler class $e_0(P)=\mathit{image}(N)=1=\mathit{image}(M)=(I)$. So, by proposition [BRS2, Proposition 6.4], there is a projective $A_n$-module $Q$ of rank $n$ and a surjective map $f:Q\twoheadrightarrow I$, and also $[P]=[Q]\in K_0(A_n)$. Fix an orientation $\chi_0:A_n\xrightarrow{\sim}\det Q$. Using an isomorphism $\gamma:F/IF\xrightarrow{\sim}Q/IQ$, with $\det\gamma\equiv\chi_0$, $f$ induces orientations $\eta:F/IF\to I/I^2$ and $\eta_i:F/m_iF\to m_i/m_i^2$, for $i=1,\dots,M$. Then, by definition,

$$e(Q,\chi_0)=(I,\eta)=\sum_{i=1}^{M}(m_i,\eta_i).$$

We can assume that

$$(m_i,\eta_i)=1\quad\forall i\leqslant s,\quad\text{and}\quad (m_i,\eta_i)=-1\quad\forall i>s.$$

Pick $u\in A_n$ such that $u-1\in m_i$ for $i\leqslant s$ and $u+1\in m_i$ for $i>s$. Let $Q'$ be defined by

$$0\to Q'\to A_n\oplus Q\xrightarrow{(u,-f)}A_n\to 0.$$

Then, by [BRS2, Lemma 5.1], $Q'$ has an orientation $\chi'$ such that

$$e(Q',\chi')=\bigl(I,\bar u^{-(n-1)}\eta\bigr)=\sum_{i=1}^{s}\bigl(m_i,\bar u^{-(n-1)}\eta_i\bigr)+\sum_{i=s+1}^{N}\bigl(m_i,\bar u^{-(n-1)}\eta_i\bigr).$$

Here $n$ is even. So, $e(Q',\chi')=\sum_{i=1}^{s}(m_i,\eta_i)+\sum_{i=s+1}^{N}(m_i,-\eta_i)=M$. This completes the proof. □

Before we proceed, we need the following proposition that relates top Chern classes with Stiefel–Whitney classes.

**Proposition 6.3.** *Let $A_n$ be the ring of algebraic functions on the real sphere $\mathbb{S}^n$ and $X = Spec(A_n)$. Then, the following diagram*

$$\begin{tikzcd} K_0(X) \arrow[r] \arrow[d, "C_0"'] & KO(\mathbb{S}^n) \arrow[d, "w_n"] \\ CH_0(X) \arrow[r] & H^n(\mathbb{S}^n, \mathbb{Z}/(2)) \end{tikzcd}$$

*commutes, where $C_0$ denotes the top Chern class map and $w_n$ denotes the top Stiefel–Whitney class.*

**Proof.** Note, $CH_0(\mathbb{R}(X)) = \mathbb{Z}/(2)$ and $H^n(\mathbb{S}^n, \mathbb{Z}/(2)) = \mathbb{Z}/(2)$. Any element in $\widetilde{K_0}(X)$ can be written as $[P] - [A_n^n]$, where $P$ is a projective $A_n$-module of rank $n$. By Bertini's theorem (see [BRS1, 2.11]), we can find a surjective map $f : P \to I$ where $I = m_1 \cap m_2 \cap \cdots \cap m_N$ is intersection of $N$ distinct maximal ideals. Assume $m_1, \ldots, m_r$ are real maximal ideals and $m_{r+1}, \ldots, m_N$ are the complex maximal ideals. For $i = 1, \ldots, r$, let $y_i \in \mathbb{S}^n$ be the point corresponding to $m_i$. So, $C_0(P) = \bar{r} \in \mathbb{Z}/(2)$, where $\bar{r}$ is the image of $r$ in $\mathbb{Z}/(2)$.

Let $\widetilde{P}$ denote the bundle on $\mathbb{S}^n$ induced by $P$. Then $f$ induces a section $s$ on the bundle $\widetilde{P}$, transversally intersecting the zerosection, exactly on the points $y_1, \ldots, y_r$. So, $w_n(\widetilde{P}) = \bar{r}$. The proof is complete. □

**Remark.** In a subsequent paper [MaSh], a more general version of Proposition 6.3 was proved later.

Now, we have the following corollary to Theorem 6.2.

**Corollary 6.4.** *Let $A_n$ be the ring of algebraic functions on $\mathbb{S}^n$. Assume $n = \dim A_n \geqslant 2$ is* ***even****. Then, the following are equivalent*:

1. *$e(P, \chi) = 1$ for some projective $A_n$-module $P$ with $rank(P) = n$ and orientation $\chi : A_n \xrightarrow{\sim} \det P$.*
2. *For some odd integer $N$, $e(P, \chi) = N$ for some projective $A_n$-module $P$ with $rank(P) = n$ and orientation $\chi : A_n \xrightarrow{\sim} \det P$.*
3. *For any odd integers $N$, $e(P, \chi) = N$ for some projective $A_n$-module $P$ with $rank(P) = n$ and orientation $\chi : A_n \xrightarrow{\sim} \det P$.*
4. *The top Chern class $C_0(P) = 1$ for some projective $A_n$-module $P$ with $rank(P) = n$.*
5. *The Stiefel–Whitney class $w_n(V) = 1$ for some vector bundle $V$ with $rank(V) = n$.*

*Let $n = 8r, 8r + 2, 8r + 4$ and let $P_n$ be a projective $A_n$-module of rank $n$ such that $[P_n] - n = \tau_n$ is the generator of $\widetilde{K_0}(A_n)$. Then above conditions are equivalent to $w_n(P_n) = 1$* (which is equivalent to *$C_0(P_n) = 1$).*

**Proof.** By (6.2), (1) ⇔ (2) ⇔ (3). It is obvious that (2) ⇔ (4). Also by (6.3), we have (4) ⇔ (5), because we can assume [Sw2] that $V$ is algebraic.

For the later part, we only need to prove that $(5) \Rightarrow w_n(P_n) = 1$. To prove this assume that $w_n(P_n) = 0$ and let $V$ be any vector bundle of rank $n$ over $\mathbb{S}^n$. We have $[V] - rank(V) = k\tau_n$. So, the total Stiefel–Whitney class $w(V) = w(k\tau_n) = w(\tau_n)^k = 1$. So, $w_n(V) = 0$. The proof is complete. □

**Remark 6.5.** We have the following summary. Let $P$ denote a projective $A_n$-module of rank $n \geqslant 2$ and $\chi : A_n \xrightarrow{\sim} \det P$ be an orientation. Then

1. For $n = 8r+3, 8r+5, 8r+7$ we have $\widetilde{K_0}(A_n) = 0$. So, the top Chern class $C_0(P) = 0$. By [BDM], $P \approx Q \oplus A_n$. Therefore $e(P, \chi) = 0$.
2. For $n = 8r+6$, we have $\widetilde{K_0}(A_n) = 0$. So, $C_0(P) = 0$, and hence $e(P, \chi)$ is always even. Further, by (6.1), for any even integer $N$ there is a projective $A_n$-module $Q$ with $rank(Q) = n$ and an orientation $\eta : A_n \xrightarrow{\sim} \det Q$, such that $e(Q, \eta) = N$.
3. For $n = 8r+1$, we have $\widetilde{K_0}(A_n) = \mathbb{Z}/2$. If $e(P, \chi)$ is even then $C_0(P) = 0$. So, $P \approx Q \oplus A_n$ and $e(P, \chi) = 0$ for all orientations $\chi$. So, only even value $e(P, \chi)$ can assume is zero.
4. Now consider the remaining cases, $n = 8r, 8r+2, 8r+4$. We have $\widetilde{K_0}(A_{8r}) = \mathbb{Z}$, $\widetilde{K_0}(A_{8r+2}) = \mathbb{Z}/2$, $\widetilde{K_0}(A_{8r+4}) = \mathbb{Z}$. As in the case of $n = 8r+6$, for any even integer $N$, for some $(Q, \eta)$ the Euler class $e(Q, \eta) = N$. The case of odd integers $N$ was discussed in (6.4).

This leads us to the following question.

**Question 6.6.** *Suppose $n = 8r, 8r+1, 8r+2, 8r+4$ and $\tau_n$ is the generator of $\widetilde{K_0}(A_n)$. Then, whether $w_n(\tau_n) = 1$ (which is equivalent to $C_0(\tau_n) = 1$)?*

Apparently, answer to this question is not known. For $n = 1$, the question has affirmative answer. We will be able to answer this question for $n = 2, 4, 8$ using the description (5.11) of the patching matrix $\psi_n$.

**Theorem 6.7.** *Let $n = 2, 4, 8$ and $A_n$ denote the ring of algebraic functions on $\mathbb{S}^n$. Let $\tau_n$ be the generator of $\widetilde{K_0}(A_n)$. Then, the top Chern class $C_0(\tau_n) = 1$.*

**Proof.** Here $n = 2, 4, 8$ and $q = q_n = \sum_{i=1}^n X_i^2$. If $M = M^0 \oplus M^1$ is an irreducible $\mathbb{Z}_2$-graded $C_n$-module, then $\tau_n = [P] - rank(P)$, where $P = \alpha(M)$ as defined in Proposition 5.5. We have $R(q + X_0^2) = A_n$ and $P$ is obtained by patching together $F^0 = (A_n)_{1+x_0} \otimes M^0$ and $F^1 = (A_n)_{1-x_0} \otimes M^1$ via $\psi = \psi_n$. In the cases of $n = 2, 4, 8$, by (5.9), $rank(P) = \dim M_0 = n$. By (5.11), with respect some bases of $M^0, M^1$, the matrix of $\psi$ has the first column $(x_1, \ldots, x_n)^t$.

We write $y = x_0$ and $F = A_n^n$. Let $e_1, \ldots, e_n$ denote the standard basis of $F$. We identify $F^0 = F_{1+y}$, $F^1 = F_{1-y}$ and consider $\psi$ as a matrix with first column $(x_1, \ldots, x_n)^t$.

Let $I = (y-1, x_1, \ldots, x_n)$ be the ideal of the north pole of $\mathbb{S}^n$. Then, $I_{1+y} = (x_1, \ldots, x_n)$. Define surjective maps $f_0 : F^0 \twoheadrightarrow I_{1+y}$ where $f_0(e_i) = x_i$ for $i = 1, \ldots, n$; and $f_1 : F^1 \twoheadrightarrow I_{1-y}$ where $f_1(e_1) = 1$ and $f_1(e_i) = x_i$ for $i = 2, \ldots, n$. We have the following patching diagram:

$$\begin{tikzcd}
P \arrow[rrr] \arrow[dr, dashed, "f"] \arrow[dd] & & & F^1 \arrow[dr, "f_1"] \arrow[dd] & \\
& I \arrow[rrr] \arrow[dd] & & & I_{1-y} \arrow[dd] \\
F^0 \arrow[rr] \arrow[dr, "f_0"] & & F^0_{1-y} \arrow[r, "\psi"] \arrow[dr, "f_0"] & F^1_{1+y} \arrow[dr, "f_1"] & \\
& I_{1+y} \arrow[rr] & & I_{1-y^2} \arrow[r, "Id"] & I_{1-y^2}.
\end{tikzcd}$$

The map $f$ is induced by the properties of fiber product diagrams. Since $f_0, f_1$ are surjective, so is $f$. Therefore, the top Chern class $C_0(P) = cycle(A_n/I) = 1$. Since $\tau_n = [P] - n$, we have $C_0(\tau_n) = 1$. This completes the proof. $\square$

**Corollary 6.8.** *Let $n = 2, 4, 8$. Then, given any integer $N$ there is a projective $A_n$-module $Q$ of rank $n$ and orientation $\chi : A_n \xrightarrow{\sim} \det Q$, such that the Euler class $e(Q, \chi) = N$.*

*Also, suppose $I$ is a locally complete intersection ideal of height $n$ and $\omega : (A_n/I)^n \twoheadrightarrow I/I^2$ is a surjective homomorphism. Then, there is a projective $A_n$-module $P$ of rank $n$, and orientation $\chi : A_n \xrightarrow{\sim} \det P$ and a surjective homomorphism $f : P \twoheadrightarrow I$ such that $(I, \omega)$ is induced by $(P, \chi)$.*

**Proof.** First part follows immediately from (6.1, 6.4, 6.7). The later part follows from [BRS2, Corollary 4.3]. □

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