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Intersection theory of algebraic obstructions

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ABSTRACT

Let A be a noetherian commutative ring of dimension d and L be a rank one projective A -module. For $1 \leq r \leq d$, we define obstruction groups $E^r(A, L)$. This extends the original definition due to Nori, in the case $r = d$. These groups would be called Euler class groups. In analogy to intersection theory in algebraic geometry, we define a product (intersection) $E^r(A, A) \times E^s(A, A) \rightarrow E^{r+s}(A, A)$. For a projective A -module Q of rank $n \leq d$, with an orientation $\chi : L \xrightarrow{\sim} \wedge^n Q$, we define a Chern class like homomorphism

$$w(Q, \chi) : E^{d-n}(A, L') \rightarrow E^d(A, LL'),$$

where L' is another rank one projective A -module.

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1. Introduction

In topology, there is a classical obstruction theory for vector bundles (See [13]). The germ of the obstruction theory in algebra was given by Nori, around 1990 [6,10]. For a smooth affine variety $X = \text{Spec}(A)$ over an infinite field k , with $\dim A = d \geq 2$, Nori outlined a definition of an obstruction group $E(A)$. Further, for a projective A -module P of rank d , with an orientation (isomorphism) $\chi : A \xrightarrow{\sim} \wedge^d P$, Nori outlined a definition of an obstruction class $e(P, \chi) \in E(A)$.

The definition of Nori was later extended by Bhatwadekar and Sridharan [2]. Given a noetherian commutative ring A with $\dim A = d \geq 2$ and a rank one projective A -module L , they defined an obstruction group $E(A, L)$. In addition, if $\mathbb{Q} \subseteq A$, given a projective A -module P of rank d and an orientation $\chi : L \xrightarrow{\sim} \wedge^d P$, they defined an obstruction class $e(P, \chi) \in E(A, L)$. They proved that $P \approx Q \oplus A$ if and only if $e(P, \chi) = 0$. Surpassing all expectations, a solid body of work has been accomplished regarding the obstructions for projective A -modules P with $\text{rank}(P) = d = \dim A$. Among them are [6,10,1,2,4,8,9].

The theory in topology is much advanced and complete. For a real smooth manifold M with $\dim M = d$ and a vector bundle V of rank $r \leq d$, there is an obstruction group and an obstruction class (called Whitney class) $w(V)$ in the obstruction group. If V has a nowhere vanishing section, then $w(V) = 0$. In case $\text{rank}(V) = r = d$, V has a nowhere vanishing section if and only if $w(V) = 0$.

Obstruction theory in algebra remains incomplete in this respect. At this time, barring [3], the theory is confined to the case when $\text{rank}(P) = d = \dim A$. Further, the intersection theory (see [5]) in algebraic geometry has also been fairly advanced and complete. Because of such an advanced status of these two theories, there has been expectations that the obstruction theory in algebra would have a similar advanced counter part. In this paper, we try to establish a foundation for a theory of algebraic obstructions, in analogy to the aforementioned theories in topology and algebraic geometry.

Given a commutative noetherian ring A of dimension d and a rank one projective A -module L , in this paper we give a definition of obstruction groups $E^r(A, L)$, for $r \geq 1$. These groups will be called Euler class groups. The obstruction group

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$E(A, L)$ mentioned above would be same as $E^d(A, L)$. Now, suppose Q is a projective A -module with $\text{rank}(Q) = n$ and an orientation (isomorphism) $\chi : L \xrightarrow{\sim} \wedge^n Q$. In [5], the top Chern class was defined as a homomorphism of degree n . In the same spirit, if $n \leq d - 2$ and L' is another projective A -module of rank one, we define a Whitney class homomorphism

$$w(Q, \chi) : E^{d-n}(A, L') \rightarrow E^d(A, L'L).$$

In case $\text{rank}(Q) = n = d - 1$, the same homomorphism is defined for $L' = A$. This homomorphism is compatible with the top Chern class homomorphism of Q . For $r \geq 2, s \geq 1$, we also define bilinear maps (intersection):

$$\cap : E^r(A, L) \times E^s(A, A) \rightarrow E^{r+s}(A, L).$$

For $r = 1$, the same is defined with $L = A$.

2. Euler class groups

In this section we define general Euler (obstruction) class groups of commutative noetherian rings.

Definition 2.1. Let A be a commutative noetherian ring with $\dim A = d$ and L be a rank one projective A -module. We write $F_r = L \oplus A^{r-1}$.

For an A -module M , the group of transvections of M will be denoted by $\mathcal{E}l(M)$ (see [7] for a definition).

1. A local L -orientation is a pair (I, ω) , where I is an ideal of A of height r and ω is an equivalence class of surjective homomorphisms $\omega : F_r/IF_r \rightarrow I/I^2$. The equivalence is defined by $\mathcal{E}l(F_r/IF_r)$ -maps. Sometimes, we will simply say that ω is a local L -orientation, to mean that (I, ω) is a local L -orientation. By abuse of notations, we sometimes denote the equivalence class of ω , by ω .
2. Let $\mathcal{L}^r(A, L)$ denote the set of all pairs (I, ω) , where I is an ideal of height r , such that $\text{Spec}(A/I)$ is connected and $\omega : F_r/IF_r \rightarrow I/I^2$ is a local L -orientation. Similarly, let $\mathcal{L}_0^r(A, L)$ denote the set of all ideals I of height r , such that $\text{Spec}(A/I)$ is connected and there is a surjective homomorphism $F_r/IF_r \rightarrow I/I^2$.
3. Let $G^r(A, L)$ denote the free abelian group generated by $\mathcal{L}^r(A, L)$ and $G_0^r(A, L)$ denote the free abelian group generated by $\mathcal{L}_0^r(A, L)$.
4. Suppose I is an ideal of height r and $\omega : F_r/IF_r \rightarrow I/I^2$ is a local L -orientation. By [3], there is a unique decomposition

$$I = I_1 \cap I_2 \cap \dots \cap I_k$$

such that $I_i + I_j = A$ for $i \neq j$ and $\text{Spec}(A/I_i)$ is connected. Then ω naturally induces local L -orientations $\omega_i : F_r/I_i F_r \rightarrow I_i/I_i^2$. Denote

$$(I, \omega) := \sum (I_i, \omega_i) \in G^r(A, L).$$

Similarly, we denote

$$(I) := \sum I_i \in G_0^r(A, L).$$

5. Global orientations: Let I be an ideal and $\omega : F_r/IF_r \rightarrow I/I^2$ be a local L -orientation. We say that ω is global, if there's a surjective lift Ω of ω as follows:

$$\begin{array}{ccc} F_r & \xrightarrow{\Omega} & I \\ \downarrow & & \downarrow \\ F_r/IF_r & \xrightarrow{\omega} & I/I^2 \end{array} .$$

6. Let $H^r(A, L)$ be the subgroup of $G^r(A, L)$, generated by global L -orientations. Also, let $H_0^r(A, L)$ be the subgroup of $G_0^r(A, L)$, generated by (I) such that I is surjective image of F_r .

Now define the Euler class group of codimension r cycles as

$$E^r(A, L) = \frac{G^r(A, L)}{H^r(A, L)}.$$

and the weak Euler class group of codimension r cycles as

$$E_0^r(A, L) = \frac{G_0^r(A, L)}{H_0^r(A, L)}.$$

Lemma 2.2. We use the notations as in Definition 2.1. For $r \geq 1$, there are natural surjective homomorphisms

$$\zeta^r : E^r(A, L) \rightarrow E_0^r(A, L).$$

Further, assume that A is Cohen Macaulay. For $r \geq 2$, there are natural homomorphisms

$$\eta^r : E_0^r(A, L) \rightarrow CH^r(A)$$

and there is a natural homomorphism

$$\eta^1 : E_0^1(A, A) \rightarrow CH^1(A).$$

Here $CH^r(A)$ denotes the Chow group of cycles of codimension r in $\text{Spec}(A)$.

Proof. Existence of ζ^r follows from the above definitions of $E^r(A, L)$, $E_0^r(A, L)$. Existence of η^r follows directly from [5, Lemma A.2.7.]. \square

3. Whitney class homomorphisms

In this section we establish the Whitney class homomorphism of a projective module, as follows.

Theorem 3.1. Let A be a commutative noetherian ring with $\dim A = d \geq 2$ and L, L' be two rank one projective A -modules. Suppose Q is a projective A -module with $\text{rank}(Q) = n \leq d - 2$ and $\det Q = L$. For an orientation $\chi : L \xrightarrow{\sim} \wedge^n Q$, there is a canonical homomorphism

$$w(Q, \chi) : E^{d-n}(A, L') \rightarrow E^d(A, L'L).$$

Also, for $\text{rank}(Q) = n = d - 1$, there is a canonical homomorphism

$$w(Q, \chi) : E^1(A, A) \rightarrow E^d(A, L).$$

Proof. We will write $F_k = L \oplus A^{k-1}$, $F'_k = L' \oplus A^{k-1}$. Let I be an ideal of height $d - n$ and

$$\omega : F'_{d-n}/IF'_{d-n} \rightarrow I/I^2$$

be an equivalence class of surjective homomorphisms (local L -orientation). To each such pair

$$(I, \omega) \in G^{d-n}(A, L'),$$

we will associate an element

$$w(Q, \chi) \cap (I, \omega) \in E^d(A, L'L).$$

First, we can find an ideal $\tilde{I} \subseteq A$ with $\text{height}(\tilde{I}) \geq d$ and a surjective homomorphism $\psi : Q/IQ \twoheadrightarrow \tilde{I}/I$. Let $\bar{\psi} = \psi \otimes A/\tilde{I}$ and $\gamma : F_n/\tilde{I}F_n \xrightarrow{\sim} Q/\tilde{I}Q$ be an isomorphism such that $\wedge^n \gamma = \chi \otimes A/\tilde{I}$. Let $\beta = \bar{\psi}\gamma$ and $\beta' : F_n/\tilde{I}F_n \rightarrow \tilde{I}/\tilde{I}^2$ be a lift of β . The following diagram

$$\begin{array}{ccc}
 Q/IQ & \xrightarrow{\psi} & \tilde{I}/I \\
 \downarrow & & \downarrow \\
 Q/\tilde{I}Q & \xrightarrow{\bar{\psi}} & \tilde{I}/(I + \tilde{I}^2) \\
 \uparrow \gamma \sim \chi & \nearrow \beta & \uparrow f \\
 F_n/\tilde{I}F_n & \xrightarrow{\beta'} & \tilde{I}/\tilde{I}^2
 \end{array} \quad (I)$$

commutes. Further, ω induces following

$$\begin{array}{ccc}
 F'_{d-n}/IF'_{d-n} & \xrightarrow{\omega} & I/I^2 \\
 \downarrow & & \downarrow \\
 F'_{d-n}/\tilde{I}F'_{d-n} & \xrightarrow{\bar{\omega}} & (I + \tilde{I}^2)/\tilde{I}^2 \\
 & \searrow \omega' & \downarrow \\
 & & \tilde{I}/\tilde{I}^2
 \end{array} \quad (II)$$

commutative diagram, where $\tilde{\omega} = \omega \otimes A/\tilde{I}$. Combining ω', β' we get a surjective homomorphism

$$\delta = \beta' \oplus \omega' : F_n/\tilde{I}F_n \oplus F'_{d-n}/\tilde{I}F'_{d-n} \cong \frac{F_n \oplus F'_{d-n}}{\tilde{I}(F_n \oplus F'_{d-n})} \twoheadrightarrow \tilde{I}/\tilde{I}^2.$$

To see that δ is surjective, consider the exact sequence

$$0 \longrightarrow (I + \tilde{I}^2)/\tilde{I}^2 \xrightarrow{\iota} \tilde{I}/\tilde{I}^2 \xrightarrow{f} \tilde{I}/(I + \tilde{I}^2) \longrightarrow 0.$$

Given $x \in \tilde{I}/\tilde{I}^2$, there is $z \in F_n/\tilde{I}F_n$ such that $f(x) = \beta'(z)$. Therefore, $x - \beta'(z) \in (I + \tilde{I}^2)/\tilde{I}^2$. This implies that $\omega'(z') = x - \beta'(z)$ for some $z' \in F'_{d-n}/\tilde{I}F'_{d-n}$. This shows that $\delta(z + z') = x$. This establishes that δ is surjective.

Now, let $\gamma_0 : \frac{LL' \oplus A^{d-1}}{\tilde{I}(LL' \oplus A^{d-1})} \xrightarrow{\sim} \frac{F_n \oplus F'_{d-n}}{\tilde{I}(F_n \oplus F'_{d-n})}$ be an isomorphism that is consistent with the natural isomorphism χ_0 :

$$\begin{array}{ccc} LL' & \xrightarrow{\sim} & \wedge^d (F_n \oplus F'_{d-n}) \\ & \searrow \sim & \uparrow \chi_0 \\ & & \wedge^d (LL' \oplus A^{d-1}) \end{array} \cdot$$

Let $\Delta = \delta\gamma_0 = (\beta', \omega')\gamma_0$. So, the diagram

$$\begin{array}{ccc} \frac{F_n \oplus F'_{d-n}}{\tilde{I}(F_n \oplus F'_{d-n})} & \xrightarrow{\delta} & \tilde{I}/\tilde{I}^2 \\ \uparrow \chi_0 \sim \gamma_0 & \nearrow \Delta & \\ \frac{LL' \oplus A^{d-1}}{\tilde{I}(LL' \oplus A^{d-1})} & & \end{array}$$

commutes.

We have $(\tilde{I}, \Delta) \in G^d(A, LL')$ is a local LL' -orientation. We will establish that the image of (\tilde{I}, Δ) in $E^d(A, LL')$ is independent of choices of ψ , the lift β' , the representative of ω and the choice of γ_0 .

1. Step-I: First we prove, for a fixed ψ , (\tilde{I}, Δ) in $E^d(A, LL')$ is independent of the lift β' , the representative ω and the choice of γ_0 .

(a) Suppose ω, ω_1 are equivalent local orientation. Then $\omega_1 = \omega\epsilon$ for some $\epsilon \in \mathcal{E}l(F'_{d-n}/IF'_{d-n})$. Using the canonical homomorphisms

$$\mathcal{E}l(F'_{d-n}/IF'_{d-n}) \rightarrow \mathcal{E}l(F'_{d-n}/\tilde{I}F'_{d-n}) \rightarrow \mathcal{E}l\left(F_n \oplus F'_{d-n}/\tilde{I}(F_n \oplus F'_{d-n})\right),$$

we have $\omega'_1 = \omega'\tilde{\epsilon}$. With $\delta_1 = (\beta', \omega'_1)$, we have

$$\delta_1 = \delta\tilde{\epsilon} \quad \text{where } \tilde{\epsilon} \in \mathcal{E}l\left(F_n \oplus F'_{d-n}/\tilde{I}(F_n \oplus F'_{d-n})\right).$$

Since

$$\epsilon_1 = \gamma_0^{-1}\tilde{\epsilon}\gamma_0 \in \mathcal{E}l\left((LL' \oplus A^{d-1})/\tilde{I}(LL' \oplus A^{d-1})\right),$$

we have

$$\Delta_1 = \delta_1\gamma_0 = \delta\tilde{\epsilon}\gamma_0 = \Delta\epsilon_1.$$

(b) Also, two different lifts β' of β differ in $(I + \tilde{I}^2)/\tilde{I}^2$ and would lead to two different Δ s that differ by $\mathcal{E}l\left(\frac{LL' \oplus A^{d-1}}{\tilde{I}(LL' \oplus A^{d-1})}\right)$.

Proof. Let $\beta'' : F_n/\tilde{I}F_n \rightarrow \tilde{I}/\tilde{I}^2$ be another lift of β . Then

$$\phi = \beta' - \beta'' : F_n/\tilde{I}F_n \rightarrow (I + \tilde{I}^2)/\tilde{I}^2.$$

We have a lift g so that the diagram

$$\begin{array}{ccc}
 F_n/\tilde{I}F_n & \xrightarrow{\phi} & (I + \tilde{I}^2)/\tilde{I}^2 \\
 \downarrow g & \nearrow \bar{\omega}_1 & \parallel \\
 F'_{d-n}/\tilde{I}F'_{d-n} & \xrightarrow{\bar{\omega}_1} & (I + \tilde{I}^2)/\tilde{I}^2 \\
 & \searrow \omega'_1 & \downarrow \\
 & & \tilde{I}/\tilde{I}^2
 \end{array}$$

commutes. Let

$$\epsilon_2 = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} : F_n/\tilde{I}F_n \oplus F'_{d-n}/\tilde{I}F'_{d-n} \xrightarrow{\sim} F_n/\tilde{I}F_n \oplus F'_{d-n}/\tilde{I}F'_{d-n}$$

It follows $(\beta'' \oplus \omega'_1)\epsilon_2 = (\beta' \oplus \omega'_1)$. Therefore, with $\Delta_2 = (\beta'' \oplus \omega'_1)\gamma_0$ and $\epsilon_3 = \gamma_0^{-1}\epsilon_2\gamma_0$, we have

$$\Delta_1 = (\beta' \oplus \omega'_1)\gamma_0 = (\beta'' \oplus \omega'_1)\epsilon_2\gamma_0 = (\beta'' \oplus \omega'_1)\gamma_0(\gamma_0^{-1}\epsilon_2\gamma_0) = \Delta_2\epsilon_3.$$

Therefore,

$$\Delta = \Delta_1\epsilon_1^{-1} = \Delta_2\epsilon_3\epsilon_1^{-1}.$$

(c) Again, if $\gamma_0 = \gamma'_0\epsilon_4$ for some $\epsilon_4 \in \mathcal{E}l\left(\frac{LL' \oplus A^{d-1}}{I(LL' \oplus A^{d-1})}\right)$ then

$$\Delta = \Delta_2\epsilon_3\epsilon_1^{-1} = (\beta'' \oplus \omega'_1)\gamma_0\epsilon_3\epsilon_1^{-1} = (\beta'' \oplus \omega'_1)\gamma'_0\epsilon_4\epsilon_3\epsilon_1^{-1} \sim (\beta'' \oplus \omega'_1)\gamma'_0.$$

So, the claim in Step-I is established. \square

2. Also note, (\tilde{I}, Δ) is independent of the choice γ . This is because, two different choices of γ would differ by a $SL(F_n/\tilde{I}F_n)$ -map. This will lead to choices of β' that will differ by a $SL(F_n/\tilde{I}F_n)$ -map.

3. Step-II: Now, we prove that $(\tilde{I}, \Delta) \in E^d(A, LL')$ is also independent of ψ . That means, it depends only on (I, ω) .

(a) To see this, first we fix a surjective lift Ω of ω as follows:

$$\begin{array}{ccc}
 F'_{d-n} & \xrightarrow{\Omega} & I \cap K \\
 \downarrow & & \downarrow \\
 F'_{d-n}/IF'_{d-n} & \xrightarrow{\omega} & I/I^2
 \end{array}$$

where $K + I = A$ and K is an ideal of height $d - n$ (or $K = A$).

(b) We can find an ideal \tilde{K} with $height(\tilde{K}) \geq d$, a surjective homomorphism ψ' and replicate the diagram (1):

$$\begin{array}{ccc}
 Q/KQ & \xrightarrow{\psi'} & \tilde{K}/K \\
 \downarrow & & \downarrow \\
 Q/\tilde{K}Q & \xrightarrow{\quad} & \tilde{K}/(K + \tilde{K}^2) \\
 \uparrow \gamma' \sim \chi & \nearrow \eta & \uparrow f' \\
 F_n/\tilde{K}F_n & \xrightarrow{\eta'} & \tilde{K}/\tilde{K}^2
 \end{array}$$

Here γ' is an isomorphism so that $\wedge^n \gamma' = \chi \otimes A/\tilde{K}$ and η' is a lift of $\eta = (\psi' \otimes A/\tilde{K})\gamma'$.

(c) Let $\omega'' : F'_{d-n}/KF'_{d-n} \rightarrow K/K^2$ be induced by Ω .

(d) As above, let $\gamma'_0 : \frac{LL' \oplus A^{d-1}}{\tilde{K}(LL' \oplus A^{d-1})} \xrightarrow{\sim} \frac{F_n \oplus F'_{d-n}}{\tilde{K}(F_n \oplus F'_{d-n})}$ be an isomorphism consistent with the canonical isomorphism $\chi_0 :$

$$\wedge^d(F_n \oplus F'_{d-n}) \xrightarrow{\sim} \wedge^d(LL' \oplus A^{d-1}).$$

(e) Combining ω'', η' we get a surjective homomorphism

$$\delta' = \eta' \oplus \omega'' : F_n/\tilde{K}F_n \oplus F'_{d-n}/\tilde{K}F'_{d-n} = \frac{F_n \oplus F'_{d-n}}{\tilde{K}(F_n \oplus F'_{d-n})} \rightarrow \tilde{K}/\tilde{K}^2.$$

(f) Write $\Delta' = \delta'\gamma'_0$. We fix (\tilde{K}, Δ') and prove that

$$(\tilde{I}, \Delta) + (\tilde{K}, \Delta') = 0 \in E^d(A, LL').$$

(g) Since $I + K \subseteq \tilde{I} + \tilde{K}$, we have $\tilde{I} + \tilde{K} = A$.

(h) We have

$$\Psi = \psi \oplus \psi' : \frac{Q}{(I \cap K)Q} \approx \frac{Q}{IQ} \oplus \frac{Q}{KQ} \rightarrow \frac{\tilde{I}}{I} \oplus \frac{\tilde{K}}{K} \approx \frac{\tilde{I} \cap \tilde{K}}{I \cap K}.$$

(i) We lift $\Psi = \psi \oplus \psi'$ to $\tilde{\Psi}$ as follows:

$$\begin{array}{ccc} Q & \xrightarrow{\tilde{\Psi}} & \tilde{I} \cap \tilde{K} \\ \downarrow & & \downarrow \\ \frac{Q}{(I \cap K)Q} & \xrightarrow{\Psi} & \frac{\tilde{I} \cap \tilde{K}}{I \cap K}. \end{array}$$

If we reduce this diagram modulo \tilde{I} or \tilde{K} , we get the following

$$\begin{array}{ccc} Q/\tilde{I}Q & \xrightarrow{\hat{\Psi}} & \tilde{I}/\tilde{I}^2 \\ \parallel & & \downarrow f \\ Q/\tilde{I}Q & \xrightarrow{\tilde{\Psi}} & \frac{\tilde{I}}{I + \tilde{I}^2} \\ \uparrow \gamma \sim \chi & \nearrow \beta & \uparrow f \\ F_n/\tilde{I}F_n & \xrightarrow{\beta'} & \tilde{I}/\tilde{I}^2 \end{array} \quad \text{and} \quad \begin{array}{ccc} Q/\tilde{K}Q & \xrightarrow{\hat{\Psi}'} & \tilde{K}/\tilde{K}^2 \\ \parallel & & \downarrow f' \\ Q/\tilde{K}Q & \xrightarrow{\tilde{\Psi}'} & \frac{\tilde{K}}{I + \tilde{K}^2} \\ \uparrow \gamma' \sim \chi & \nearrow \eta & \uparrow f' \\ F_n/\tilde{K}F_n & \xrightarrow{\eta'} & \tilde{K}/\tilde{K}^2 \end{array}$$

commutative diagrams. It follows that

$$\alpha_1 = \beta'\gamma^{-1} - \hat{\Psi} : Q/\tilde{I}Q \rightarrow \frac{I + \tilde{I}^2}{\tilde{I}^2} \subseteq \frac{\tilde{I}}{\tilde{I}^2}.$$

Similarly, we have

$$\alpha_2 = \eta'(\gamma')^{-1} - \hat{\Psi}' : Q/\tilde{K}Q \rightarrow \frac{K + \tilde{K}^2}{\tilde{K}^2} \subseteq \frac{\tilde{K}}{\tilde{K}^2}.$$

We lift α_1, α_2 to g_1, g_2 so that the diagrams

$$\begin{array}{ccc} Q/\tilde{I}Q & \xrightarrow{\alpha_1} & (I + \tilde{I}^2)/\tilde{I}^2 \\ \downarrow g_1 & \nearrow & \parallel \\ F'_{d-n}/\tilde{I}F'_{d-n} & \xrightarrow{\tilde{\omega}} & (I + \tilde{I}^2)/\tilde{I}^2, \\ & \searrow \omega' & \downarrow \\ & & \tilde{I}/\tilde{I}^2 \end{array} \quad , \quad \begin{array}{ccc} Q/\tilde{K}Q & \xrightarrow{\alpha_2} & (K + \tilde{K}^2)/\tilde{K}^2 \\ \downarrow g_2 & \nearrow & \parallel \\ F'_{d-n}/\tilde{K}F'_{d-n} & \xrightarrow{\tilde{\omega}'} & (K + \tilde{K}^2)/\tilde{K}^2 \\ & \searrow \omega'' & \downarrow \\ & & \tilde{K}/\tilde{K}^2 \end{array}$$

commute.

(j) Let g be given by g_1, g_2 and $\tilde{\gamma}$ be given by γ, γ' . Write

$$\Gamma = \begin{pmatrix} 1 & 0 \\ g & 1 \end{pmatrix} \begin{pmatrix} \tilde{\gamma} & 0 \\ 0 & 1 \end{pmatrix} : \begin{array}{ccc} \frac{F_n \oplus F'_{d-n}}{(\tilde{I} \cap \tilde{K})(F_n \oplus F'_{d-n})} & \xrightarrow{\quad} & \frac{Q \oplus F'_{d-n}}{(\tilde{I} \cap \tilde{K})(Q \oplus F'_{d-n})} \\ & \searrow \Gamma & \downarrow \\ & & \frac{Q \oplus F'_{d-n}}{(\tilde{I} \cap \tilde{K})(Q \oplus F'_{d-n})} \end{array} .$$

(k) Consider the surjective homomorphism

$$Q \oplus F'_{d-n} \xrightarrow{\tilde{\psi} \oplus \bar{\Omega}} \tilde{I} \cap \tilde{K}.$$

(l) We claim that the diagram

$$\begin{array}{ccc} Q \oplus F'_{d-n} & \xrightarrow{\tilde{\psi} \oplus \bar{\Omega}} & \tilde{I} \cap \tilde{K} \\ \downarrow & & \downarrow \\ \frac{Q \oplus F'_{d-n}}{(\tilde{I} \cap \tilde{K})(Q \oplus F'_{d-n})} & \xrightarrow{\hat{\psi} \oplus \bar{\Omega}} & \frac{\tilde{I} \cap \tilde{K}}{(\tilde{I} \cap \tilde{K})^2} \\ \uparrow \Gamma & \nearrow \delta, \delta' & \\ \frac{F_n \oplus F'_{d-n}}{(\tilde{I} \cap \tilde{K})(F_n \oplus F'_{d-n})} & & \end{array}$$

commutes. Only the commutativity of the bottom triangle needs to be checked. This is done by checking on $V(\tilde{I}), V(\tilde{K})$ separately. We check directly that

$$\left(\hat{\psi} \oplus \bar{\Omega} \right) \Gamma(0, y) = \left(\hat{\psi} \oplus \bar{\Omega} \right) (0, y) = \bar{\Omega}(y) = (\delta, \delta')(0, y)$$

and

$$\begin{aligned} \left(\hat{\psi} \oplus \bar{\Omega} \right) \Gamma(x, 0) &= \left(\hat{\psi} \oplus \bar{\Omega} \right) (\tilde{\gamma}(x), g\tilde{\gamma}(x)) \\ &= \hat{\psi} \tilde{\gamma}(x) + \bar{\Omega} g\tilde{\gamma}(x) = \hat{\psi} \tilde{\gamma}(x) + \beta' \gamma^{-1} \gamma(x) - \hat{\psi} \tilde{\gamma}(x) \beta'(x) = (\delta, \delta')(x, 0). \end{aligned}$$

(m) Composing with (γ_0, γ'_0) , it follows from [2, Cor. 4.4] that (Δ, Δ') is global. Therefore,

$$\left(\tilde{I}, \Delta \right) + \left(\tilde{K}, \Delta' \right) = 0 \in E^d(A, LL').$$

(n) Since, (\tilde{K}, Δ') depends only on (I, ω) , it follows (\tilde{I}, Δ) is independent of choice of ψ . This establishes the claim in Step-II.

4. If (I, ω) is global, then in the above proof, we can take $K = A$ and it follows (\tilde{I}, Δ) is global.

Now, the association

$$(I, \omega) \mapsto \left(\tilde{I}, \Delta \right) \in E^d(A, LL')$$

defines a homomorphism

$$\varphi(Q, \chi) : G^{d-n}(A, L') \rightarrow E^d(A, LL')$$

where $(I, \omega) \in G^{d-n}(A, L')$ are the free generators of $G^{d-n}(A, L')$ and $(\tilde{I}, \Delta) \in E^d(A, LL')$ are as above. The above discussions establish that $\varphi(Q, \chi)$ is a well-defined homomorphism.

By (4), it follows that φ factors through a homomorphism

$$w(Q, \chi) : E^{d-n}(A, L') \rightarrow E^d(A, LL').$$

This completes the proof of the theorem. \square

By forgetting the orientation in Theorem 3.1, we have the following for weak Euler class groups.

Corollary 3.2. Let A, L, L', Q be as in [Theorem 3.1](#). If $\text{rank}(Q) \leq d - 2$, then there is a canonical homomorphism

$$w_0(Q) : E_0^{d-n}(A, L') \rightarrow E_0^d(A, L'L) \approx E_0^d(A, A).$$

Also for $\text{rank}(Q) = n = d - 1$, there is a canonical homomorphism

$$w_0(Q) : E_0^1(A, A) \rightarrow E_0^d(A, L) \approx E_0^d(A, A).$$

Proof. The isomorphisms at the right side were proved in [2]. The rest of the proof is similar to that of [Theorem 3.1](#) and we give an outline. We will write $F_k = L \oplus A^{k-1}$, $F'_k = L' \oplus A^{k-1}$.

Suppose (I) is a generator of $G_0^{d-n}(A, L')$. Here I is an ideal of height $d - r$, $\text{Spec}(A/I)$ is connected and there is a surjective homomorphism $\frac{F'}{IF'_n} \rightarrow I/I^2$. There is a surjective homomorphism $\psi : Q/IQ \rightarrow \tilde{I}/I$. For such a generator (I) we associate $(\tilde{I}) \in E_0^d(A, LL')$.

We fix a local orientation, $\omega : \frac{F'}{IF'_n} \rightarrow I/I^2$, and a surjective lift $\Omega : F' \rightarrow I \cap K$ of ω , where K is an ideal of height $d - n$. Now, let $\psi' : Q/KQ \rightarrow \tilde{K}/K$, be a surjective homomorphism, where \tilde{K} is an ideal of height d . As in [Theorem 3.1](#), we prove that there is a surjective homomorphism $LL' \oplus A^{d-1} \rightarrow \tilde{I} \cap \tilde{K}$. This shows that

$$(\tilde{I}) + (\tilde{K}) = 0 \in E_0^d(A, LL')$$

and so $(\tilde{I}) \in E_0^d(A, LL')$ is independent of choice of ψ .

This association $(I) \mapsto (\tilde{I}) \in E_0^d(A, LL')$, extends to homomorphism

$$\varphi_0 : G_0^{d-n}(A, L') \rightarrow E_0^d(A, LL').$$

If (I) is global, i.e. I is surjective image of F'_{d-n} , taking $K = A$ in the above argument, we prove (\tilde{I}) is global. So, φ_0 factors through a homomorphism

$$w_0(Q) : E_0^{d-n}(A, L') \rightarrow E_0^d(A, LL').$$

This completes the proof. \square

Definition 3.3. This homomorphism $w(Q, \chi)$ in [Theorem 3.1](#), will be called the *Whitney class homomorphism*. The image of $(I, \omega) \in E^{d-n}(A, L')$ under $w(Q, \chi)$ will be denoted by $w(Q, \chi) \cap (I, \omega)$.

Similarly, the homomorphism $w_0(Q)$ in (3.2) will be called the *weak Whitney class homomorphism*. The image of $(I) \in E_0^{d-n}(A, L')$ under $w_0(Q)$ will be denoted by $w_0(Q) \cap (I)$.

The following is about the compatibility of these homomorphisms, along with the Chern class homomorphisms.

Corollary 3.4. We use the notations as in [Theorem 3.1](#) and [Corollary 3.2](#). Further assume that A is a Cohen–Macaulay ring. Then, we have

$$w_0(Q)\zeta^{d-n} = \zeta^d w(Q, \chi) \quad \text{and} \quad C^n(Q^*)\eta^{d-n} = \eta^d w_0(Q),$$

where $\zeta^{d-n}, \zeta^d, \eta^{d-n}, \eta^d$ are the natural homomorphisms as in [Lemma 2.2](#) and $C^n(Q^*)$ denotes the top Chern class homomorphism ([5]) of Q^* .

Proof. The first identity follows from the definitions of $w_0(Q)$ and $w(Q, \chi)$. To prove the second identity, let I be an ideal of height $d - n$ with a surjective homomorphism $\frac{L' \oplus A^{d-n-1}}{I(L' \oplus A^{d-n-1})} \rightarrow I/I^2$. By Eisenbud–Evans theorem (see [12]), we can find a surjective homomorphism $\psi : Q \rightarrow J$, where J is an ideal of $\text{height}(J) = n$ and $\text{height}(I + J) = d$. Write $\tilde{I} = I + J$. Note that I, J, \tilde{I} are locally complete intersection ideals. We have,

$$\eta^d w_0(Q) \cap (I) = \eta^d(\tilde{I}) = \text{cycle}(A/\tilde{I}).$$

Also

$$C^n(Q^*)\eta^{d-n}(I) = C^n(Q^*) \cap \text{cycle}(A/I).$$

We have

$$C^n((Q/IQ)^*) \cap (\text{cycle}(A/I)) = (\text{cycle}(A/\tilde{I})) \in CH^n(A/I).$$

With $f : \text{Spec}(A/I) \subseteq \text{Spec}(A)$, apply f_* . By the projection formula [5], we have

$$C^n(Q^*) \cap (\text{cycle}(A/I)) = (\text{cycle}(A/\tilde{I})) \in CH^d(A).$$

So, the proof is complete. \square

The following is about vanishing of Whitney class homomorphisms.

Theorem 3.5. Suppose $A, L, L', Q, \chi, F_k, F'_k$ be as in Theorem 3.1 and its proof. Let I be an ideal of height $d-n$ and $\omega : \frac{F'_{d-n}}{IF'_{d-n}} \rightarrow I/I^2$ be a local L' -orientation. If $Q/IQ = P_0 \oplus A/I$, then

$$w(Q, \chi) \cap (I, \omega) = 0 \in E^d(A, LL').$$

In particular, if $Q = P \oplus A$, then the homomorphism

$$w(Q, \chi) : E^{d-n}(A, L') \rightarrow E^d(A, LL')$$

is identically zero. Similar statements holds for $w_0(Q)$.

Proof. Proof of the last statement follows from the former assertions, by (3.4). We will enumerate the steps of the proof of the former assertions.

1. By Eisenbud–Evans theorem [12], there is an ideal $\tilde{I} \subseteq A$ with $\text{height}(\tilde{I}) = d$ and a surjective homomorphism $\psi : Q/IQ \rightarrow \tilde{I}/I$. The following diagram

$$\begin{array}{ccc}
 Q/IQ & \xrightarrow{\psi} & \tilde{I}/I \\
 \downarrow & & \downarrow \\
 Q/\tilde{I}Q & \xrightarrow{\bar{\psi}} & \tilde{I}/(I + \tilde{I}^2) \\
 \uparrow \gamma \sim \chi & \nearrow \beta & \uparrow f \\
 F_n/\tilde{I}F_n & \xrightarrow{\beta'} & \tilde{I}/\tilde{I}^2
 \end{array} \quad (I)$$

commutes. Here $\bar{\psi} = \psi \otimes A/\tilde{I}$ and γ is any isomorphism, with $\wedge^n \gamma = \chi \otimes A/\tilde{I}$, $\beta = \bar{\psi} \gamma$ and β' is a lift of β .

2. Let Ω be any lift of ω . Then, the following diagram

$$\begin{array}{ccc}
 F'_{d-n} & \xrightarrow{\Omega} & I \\
 \downarrow & & \downarrow \\
 \frac{F'_{d-n}}{IF'_{d-n}} & \xrightarrow{\omega} & I/I^2 \\
 \downarrow & & \downarrow \\
 \frac{F'_{d-n}}{\tilde{I}F'_{d-n}} & \xrightarrow{\bar{\omega}} & I/\tilde{I} \\
 \searrow \omega' & & \downarrow \\
 & & \tilde{I}/\tilde{I}^2
 \end{array}$$

commutes. Here $\bar{\omega} = \omega \otimes A/\tilde{I}$.

3. With $Q/IQ = P_0 \oplus A/I$, let θ be the restriction of ψ to P_0 . We can write $\psi = (\theta, \bar{a})$ for some $a \in \tilde{I}$. Write $\psi(P_0) = \tilde{J}/I$, for some ideal \tilde{J} containing I . In fact, $\text{height}(\tilde{J}) = d - 1$ and $\tilde{I} = (\tilde{J}, a)$.
4. Since $\text{height}(\tilde{J}) = d - 1$, there is an isomorphism

$$\gamma' : (F_{n-1}/\tilde{J}F_{n-1}) \xrightarrow{\sim} P_0/\tilde{J}P_0.$$

5. Define χ' as in the commutative diagram:

$$\begin{array}{ccc}
 L/\tilde{J}L & \xrightarrow{\chi \otimes A/\tilde{J}} & \wedge^n \left(\frac{P_0 \oplus A/I}{\tilde{J}(P_0 \oplus A/I)} \right) \\
 \searrow \chi' & & \uparrow \wr \\
 & & \wedge^{n-1} P_0/\tilde{J}P_0
 \end{array}$$

By adjusting the determinant, if needed, we can assume that $\wedge^{n-1} \gamma' = \chi'$.

6. For our purpose, γ is a choice. So, we can assume that γ is the reduction of $\gamma' \oplus 1$.
7. A commutative diagram similar to (I) is induced by θ as follows:

$$\begin{array}{ccc}
 P_0 & \xrightarrow{\theta} & \tilde{J}/I \\
 \downarrow & & \downarrow \\
 P_0/\tilde{J}P_0 & \xrightarrow{\bar{\theta}} & \tilde{J}/(I + \tilde{J}^2) \\
 \uparrow \zeta & \nearrow \xi & \uparrow \\
 F_{n-1}/\tilde{J}F_{n-1} & \xrightarrow{\zeta'} & \tilde{J}/\tilde{J}^2
 \end{array} \quad (II).$$

Here $\bar{\theta} = \theta \otimes A/\tilde{J}$ and ζ' is a lift of $\zeta = \bar{\theta}\gamma'$. If we tensor this diagram with A/\tilde{I} , we get the following commutative diagram:

$$\begin{array}{ccccc}
 P_0/\tilde{I}P_0 & \xrightarrow{\theta} & \tilde{J}/(I + \tilde{J}\tilde{I}) & & \\
 \parallel & & \parallel & & \\
 P_0/\tilde{I}P_0 & \xrightarrow{\bar{\theta}} & \tilde{J}/(I + \tilde{J}\tilde{I}) & \xrightarrow{\quad} & \tilde{I}/(I + \tilde{I}^2) \\
 \uparrow \zeta & \nearrow \bar{\xi} & \uparrow \beta_{F_{n-1}} & \nearrow & \uparrow f \\
 F_{n-1}/\tilde{I}F_{n-1} & \xrightarrow{\bar{\zeta}'} & \tilde{J}/\tilde{J}\tilde{I} & \xrightarrow{\quad} & \tilde{I}/\tilde{I}^2
 \end{array} \quad (III).$$

8. This shows that $(\bar{\zeta}', \bar{a})$ is a lift of β . Since $w(Q, \chi) \cap (I, \omega)$ is independent of the lift β' we can assume that $\beta' = (\bar{\zeta}', \bar{a})$.

9. We lift ζ' to a homomorphism $\delta : F_{n-1} \rightarrow \tilde{J}$.
10. The homomorphism

$$(\delta, a, \Omega) : F_{n-1} \oplus A \oplus F'_{d-n} \rightarrow \tilde{I}$$

lifts $(\beta', \omega') = (\bar{\zeta}', \bar{a}, \omega')$.

11. Let $\tilde{J}' = \delta(F_{n-1}) + \Omega(F'_{d-n})$. Then

$$\tilde{J} = \tilde{J}' + \tilde{J}^2.$$

To see this, let $y \in \tilde{J}$. Then, from diagram (II), there is $x \in F_{n-1}$ such that

$$\delta(x) - y = y_1 + z \quad \text{where } y_1 \in I, z \in \tilde{J}^2$$

Now, there is $x_1 \in F'_{d-n}$ such that

$$y_1 - \Omega(x_1) = z_1 \in I^2 \subseteq \tilde{J}^2.$$

Therefore $\delta(x) - \Omega(x_1) - y = z + z_1$.

12. There is an isomorphism

$$\gamma''_0 : \frac{LL' \oplus A^{d-2}}{\tilde{J}(LL' \oplus A^{d-2})} \xrightarrow{\sim} \frac{F_{n-1} \oplus F'_{d-n}}{\tilde{J}(F_{n-1} \oplus F'_{d-n})}$$

such that the determinant is given by the commutative diagram:

$$\begin{array}{ccc}
 LL'/\tilde{J}LL' & \xrightarrow{\sim} & \frac{LL' \oplus A^{d-2}}{\tilde{J}(LL' \oplus A^{d-2})} \\
 \searrow \sim & & \downarrow \wr^{\wedge^{d-1} \gamma''_0} \\
 & & \frac{F_{n-1} \oplus F'_{d-n}}{\tilde{J}(F_{n-1} \oplus F'_{d-n})}
 \end{array}$$

13. Let

$$\Gamma_0'' : LL' \oplus A^{d-2} \rightarrow F_{n-1} \oplus F'_{d-n}$$

be any lift of γ_0'' and

$$\gamma_0 = (\gamma_0'' \oplus 1) \otimes A/\tilde{I} : \frac{LL' \oplus A^{d-1}}{\tilde{I}(LL' \oplus A^{d-1})} \xrightarrow{\sim} \frac{F_n \oplus F'_{d-n}}{\tilde{I}(F_n \oplus F'_{d-n})}.$$

14. Let $\Gamma = (\delta, \Omega)\Gamma_0''$. Write $\tilde{J}'' = \Gamma(LL' \oplus A^{d-2})$. By (6),

$$\tilde{J} = \tilde{J}'' + \tilde{J}^2.$$

15. There is an element $\epsilon \in \tilde{J}^2$ such that $(1 - \epsilon)\tilde{J} \subseteq \tilde{J}''$. So, as in [11], we have

$$\tilde{J} = (\tilde{J}'', \epsilon) \quad \text{and} \quad \tilde{I} = (\tilde{J}, a) = (\tilde{J}'', \epsilon + (1 - \epsilon)a).$$

16. It follows, with $b = \epsilon + (1 - \epsilon)a$, that

$$(\Gamma, b) : (LL' \oplus A^{d-2}) \oplus A \rightarrow \tilde{I}$$

is a surjective homomorphism.

17. Now, we have

$$(\Gamma, b) \otimes A/\tilde{I} = ((\delta, \Omega)\Gamma_0, b) \otimes A/\tilde{I} = (\delta, \Omega, b)(\Gamma_0 \oplus 1) \otimes A/\tilde{I}$$

which is

$$= ((\delta, \Omega, b) \otimes A/\tilde{I}) \gamma_0 = (\beta', \omega') \gamma_0 \sim \Delta.$$

The last equality follows from (6). Therefore Δ is global.

Therefore,

$$w(Q, \chi) \cap (I, \omega) = (\tilde{I}, \Delta) = 0 \in E(A, LL').$$

The proof is complete. \square

In analogy to results in Chern class theory [5], we have the following.

Corollary 3.6. *Let A be a noetherian commutative ring with $\dim A = d \geq 2$. Suppose $L, L_1, L_2 \in \text{Pic}(A)$. Then, for $x \in E_0^{d-1}(A, L)$, we have*

$$w_0(L_1 L_2) \cap x = w_0(L_1) \cap x + w_0(L_2) \cap x \in E_0^d(A, L) \approx E_0^d(A, A).$$

Proof. We can assume $x = (I)$ where I is an ideal of height $d - 1$ and there is a surjective homomorphism $\frac{L \oplus A^{d-2}}{I(L \oplus A^{d-2})} \rightarrow I/I^2$.

There are ideals \tilde{I}_1, \tilde{I}_2 of height d and surjective homomorphisms $L_i/IL_i \rightarrow \tilde{I}_i/I \subseteq A/I$, for $i = 1, 2$. Since, $\dim A/I \leq 1$, we can assume that $\tilde{I}_1 + \tilde{I}_2 = A$. Therefore,

$$L_1 L_2 \otimes A/I \approx L_1/IL_1 \otimes L_2/IL_2 \quad \text{maps onto} \quad \tilde{I}_1/I \otimes \tilde{I}_2/I = \tilde{I}_1 \cap \tilde{I}_2/I.$$

So,

$$w_0(L_1 L_2) \cap (I) = (\tilde{I}_1 \cap \tilde{I}_2) = (\tilde{I}_1) + (\tilde{I}_2) = w_0(L_1) \cap (I) + w_0(L_2) \cap (I).$$

The proof is complete. \square

4. Intersections in Euler class groups

We define an intersection product in the Euler class groups as follows.

Definition 4.1. Suppose A is a noetherian commutative ring with $\dim A = d \geq 2$ and L, L' be two projective A -modules of rank one. We write $F = L \oplus A^{r-1}, F' = L' \oplus A^{s-1}$. Let I be an ideal of height r and J be an ideal of height s . Assume $\text{height}(I + J) \geq r + s$ and suppose

$$\omega : F/IF \rightarrow I/I^2 \quad \text{and} \quad \omega' : F'/JF' \rightarrow J/J^2$$

are two local orientations. Then ω, ω' induce a surjective homomorphism

$$\eta : \frac{F \oplus F'}{(I + J)(F \oplus F')} \rightarrow \frac{(I + J)}{(I + J)^2}$$

according to the following

$$\begin{array}{ccc}
 \frac{F \oplus F'}{(I+J)(F \oplus F')} & \xrightarrow{(\bar{\omega} \oplus \bar{\omega}')} & \frac{I}{(I+J)I} \oplus \frac{J}{(I+J)J} \\
 & \searrow \eta & \downarrow \\
 & & \frac{(I+J)}{(I+J)^2}
 \end{array}$$

commutative diagram, where $\bar{\omega} = \omega \otimes A/(I + J)$, $\bar{\omega}' = \omega' \otimes A/(I + J)$. (We will continue to abuse notations and denote elements in $G^r(A, L)$ their images in $E^r(A, L)$ by same notations.)

1. If $L' = A$, define *intersection*

$$(I, \omega) \cap (J, \omega') := (I + J, \eta) \in E^{r+s}(A, L).$$

The right hand side is interpreted as zero, if $I + J = A$.

2. If $r + s = d$, define *intersection*

$$(I, \omega) \cap (J, \omega') := (I + J, \eta\gamma_0) \in E^{r+s}(A, LL')$$

where $\gamma_0 : \frac{LL' \oplus A^{d-1}}{(I+J)(LL' \oplus A^{d-1})} \xrightarrow{\sim} \frac{F \oplus F'}{(I+J)(F \oplus F')}$ is any isomorphism with $\wedge^d \gamma_0 = Id$. The right hand side is interpreted as zero, if $I + J = A$.

Subsequently, we prove that the above (4.1) are well defined in the general set-up. The following lemma is about intersections of global orientations.

Lemma 4.2. *With all notations as in Definition 4.1, suppose ω' is global. If $s \geq 2$ or $L' = A$. Then there is a surjective lift $\Theta : F \oplus F' \rightarrow (I + J)$ of η . In particular,*

1. in case $L' = A$, the local orientation η is global;
2. and also in case $L' \neq A$, and $r + s = d$, then $\eta\gamma_0$ is global.

Proof. If $I + J = A$, there is nothing to prove. So, we assume $height(I + J) = r + s$. Suppose $f' = (g, a) : F' \rightarrow J$ is a surjective lift of ω' . Also, let $f : F \rightarrow I$ be any lift of ω . There is an $e \in I^2$ such that $(1 - e)I \subseteq f(F)$. We claim that [11],

$$\text{with } b = e + (1 - e)a, \quad (I + J) = (f(F), g(L' \oplus A^{s-2}), b) =: \mathcal{K}.$$

First, $(1 - e) \in f(F)_e$. So, $(1 - e)a \in \mathcal{K}_e$. So, $e = b - (1 - e)a \in \mathcal{K}_e$. Therefore, $A_e = (I + J)_e = \mathcal{K}_e$. If $s > 1$, let $F'' = L' \oplus A^{s-2}$ and if $s = 1$, let $F'' = 0$. We have $\mathcal{K}_{1-e} =$

$$(f(F)_{1-e}, g(F'')_{1-e}, b) = (I_{1-e}, g(F'')_{1-e}, b) = (I_{1-e}, g(F'')_{1-e}, a) = (I + J)_{1-e}.$$

This shows that $\Theta = (f, g, b)$ is a surjective lift of η .

Now we prove (2). Write $\theta = (f, g)$ and $\mathfrak{I} = \theta(F \oplus F')$. Since $\dim A/\mathfrak{I} \leq 1$, there is an isomorphism $\gamma'_0 : \frac{LL' \oplus A^{d-2}}{\mathfrak{I}(LL' \oplus A^{d-2})} \xrightarrow{\sim} \frac{F \oplus F''}{\mathfrak{I}(F \oplus F')}$ with $\wedge^{d-1} \gamma'_0 = 1$. Since γ_0 is a choice, we can assume that $\gamma_0 = (\gamma'_0 \oplus 1) \otimes A/(I + J)$. Let $\Gamma_0 : LL' \oplus A^{d-2} \rightarrow F \oplus F''$ be any lift of γ_0 . Let $\tilde{K} = \theta \Gamma_0 (LL' \oplus A^{d-2})$. Then $\mathfrak{I} = \tilde{K} + \mathfrak{I}^2$ and $I + J = (\mathfrak{I}, b)$.

There is $\epsilon \in \mathfrak{I}^2$ such that $(1 - \epsilon)\mathfrak{I} \subseteq \tilde{K}$. Write $c = \epsilon + (1 - \epsilon)b$. Then

$$I + J = (\mathfrak{I}, b) = (\tilde{K}, c).$$

Therefore $(\theta \Gamma_0, c) : (LL' \oplus A^{d-2}) \rightarrow I + J$ is a surjective lift of $\eta\gamma_0$. So, $\eta\gamma_0$ is global. This completes the proof. \square

The following moving lemma is a tool for the rest of this paper.

Lemma 4.3. *Suppose A is noetherian commutative ring with $\dim A = d$. Let I, J be two ideals of height r and s , respectively. Suppose F is a projective A -module of rank r and $\omega : F/IF \rightarrow I/I^2$ is a surjective homomorphism. Also assume that J is locally generated by s elements. Then, there is a surjective lift $f : F \rightarrow I \cap K$ of ω such that (1) $I + K = A$ (2) $height(K) \geq r$ and (3) $height(K + J) \geq r + s$.*

Proof. The proof is done by the use of standard generalized dimension theory. First, there is a lift $f_0 : F \rightarrow I$ of ω . Then $I = (f_0(F), a)$ for some $a \in I^2$.

Let $\mathcal{P}_{r-1} \subseteq Spec(A)$, be the set of all prime ideals \wp , with $height(\wp) \leq r - 1$ and $a \notin \wp$. Also, let $\mathcal{Q}_{r-1} \subseteq Spec(A)$ be the set of all prime ideals \wp such that $J \subseteq \wp$, $a \notin \wp$ and $height(\wp/J) \leq r - 1$. Write $\mathcal{P} = \mathcal{P}_{r-1} \cup \mathcal{Q}_{r-1}$.

Let $d_1 : \mathcal{P}_{r-1} \rightarrow \mathbb{N}$ be the restriction of the usual dimension function and $d_2 : \mathcal{Q}_{r-1} \rightarrow \mathbb{N}$ be the dimension function induced by that on $Spec(A_a/J_a)$. Then d_1, d_2 induce a generalized dimension function $d : \mathcal{P} \rightarrow \mathbb{N}$, ([12] or see [7]).

Now, $(f_0, a) \in F^* \oplus A$ is a basic element on \mathcal{P} . Since, $rank(F) = r > d(\wp)$ for all $\wp \in \mathcal{P}$, there is a $g \in F^*$, such that $f = f_0 + ag$ is basic on \mathcal{P} . Clearly, f is a lift of ω and $I = (f(F), a)$.

Since, f is a lift of ω , we can write $f(F) = I \cap K$, such that $I + K = A$. It is routine to check now that $height(K) \geq r$ and $height\left(\frac{I+K}{J}\right) \geq r$. Since $\left(\frac{I+K}{J}\right)$ is locally r generated, $height\left(\frac{I+K}{J}\right) = r$ and for any minimal prime \wp over $J + K$ we have $height\left(\frac{I+K}{J}\right) = height\left(\frac{\wp}{J}\right) = r$.

Also, since J is locally s generated ideal of height s , $height(\wp) = height(J)$, for any minimal prime \wp over J .

If $J + K = A$, there is nothing to prove. Suppose $J + K \subseteq \wp^1$ be a minimal prime over $J + K$ such that $height(J + K) = height(\wp^1)$, and $J \subseteq \wp^0$ be a minimal prime over J , such that $height(\wp^1/\wp^0) = height(\wp^1/J)$. We have,

$$height(\wp^1) \geq height(\wp^1/\wp^0) + height(\wp^0) = height(\wp^1/J) + height(J)$$

$= r + s$. This completes the proof. \square

The following lemma establishes the consistency of Definition 4.1.

Lemma 4.4. We use all the notations in Definition 4.1. Assume $r \geq 2$ or $L = A$. Further, let I_1 be an ideal of height r with $height(I_1 + J) \geq r + s$ and let

$$\omega_1 : F/I_1F \rightarrow I_1/I_1^2$$

be a local orientation. Suppose $(I, \omega) = (I_1, \omega_1) \in E^r(A, L)$.

1. If $L' = A$, then

$$(I, \omega) \cap (J, \omega') = (I_1, \omega_1) \cap (J, \omega') \in E^{r+s}(A, L).$$

2. If $r + s = d$, then

$$(I, \omega) \cap (J, \omega') = (I_1, \omega_1) \cap (J, \omega') \in E^d(A, LL').$$

Proof. We will only prove (1). The proof of (2) is similar.

We have either $height(I_1 + J) = r + s$ or $I_1 + J = A$. We will assume $L' = A$. By Lemma 4.3, we can find a surjective lift $\theta : F \rightarrow I \cap K$ of ω , where $I + K = A$ and $height(K + J) \geq r + s$. Since $(I \cap K, \theta \otimes A/(I \cap K))$ is global, by Lemma 4.2, we have

$$(I, \omega) \cap (J, \omega') + (K, \bar{\theta}) \cap (J, \omega') = (I \cap K + J, \bar{\theta} \oplus \bar{\omega}') = 0 \in E^{r+s}(A, L).$$

(This also works when $K + J = A$, in this case $I \cap K + J = I + J$.) We will prove

$$(I_1, \omega_1) \cap (J, \omega') + (K, \bar{\theta}) \cap (J, \omega') = 0 \in E^{r+s}(A, L).$$

Since $(I, \omega) = (I_1, \omega_1) \in E^r(A, L)$ we have $(I_1, \omega_1) + (K, \bar{\theta}) = 0 \in E^r(A, L)$. There exist, ideals $\{\mathfrak{I}_t : 1 \leq t \leq i + j\}$ of height r and global orientations $\theta_t : F/\mathfrak{I}_tF \rightarrow \mathfrak{I}_t/\mathfrak{I}_t^2$ such that

$$(I_1, \omega_1) + (K, \bar{\theta}) + \sum_{t=i+1}^{i+j} (\mathfrak{I}_t, \theta_t) = \sum_{t=1}^i (\mathfrak{I}_t, \theta_t) \in G^r(A, L) \quad (I).$$

Again, by Lemma 4.3, there is a surjective lift $\Omega' : F' \rightarrow J \cap J'$ of ω' , such that $height(J') \geq s$, $J + J' = A$ and

$$height\left(J' + I \cap K \cap \bigcap_{t=1}^{i+j} \mathfrak{I}_t\right) \geq r + s.$$

The identity (I) is a purely formal identity. Note that Definition 4.1 applies to all the terms in the following expression and by formal argument, it follows that from (I) that

$$\begin{aligned} &(I_1, \omega_1) \cap (J', \bar{\Omega}') + (K, \bar{\theta}) \cap (J', \bar{\Omega}') + \sum_{t=i+1}^{i+j} (\mathfrak{I}_t, \theta_t) \cap (J', \bar{\Omega}') \\ &= \sum_{t=1}^i (\mathfrak{I}_t, \theta_t) \cap (J', \bar{\Omega}') \end{aligned}$$

in $G^{r+s}(A, L)$. By Lemma 4.2, we have

$$(I_1, \omega_1) \cap (J', \bar{\Omega}') + (K, \bar{\theta}) \cap (J', \bar{\Omega}') = 0 \in E^{r+s}(A, L) \quad (II).$$

We also have

$$\begin{aligned} &(I_1, \omega_1) \cap (J, \omega') + (K, \bar{\theta}) \cap (J, \omega') + (I_1, \omega_1) \cap (J', \bar{\Omega}') + (K, \bar{\theta}) \cap (J', \bar{\Omega}') \\ &= ((I_1, \omega_1) \cap (J, \omega') + (I_1, \omega_1) \cap (J', \bar{\Omega}')) + ((K, \bar{\theta}) \cap (J, \omega') + (K, \bar{\theta}) \cap (J', \bar{\Omega}')) \\ &= (I_1, \omega_1) \cap (J \cap J', \bar{\Omega}') + (K, \bar{\theta}) \cap (J \cap J', \bar{\Omega}') = 0. \end{aligned}$$

Combining with (II), we have

$$(I_1, \omega_1) \cap (J, \omega') + (K, \bar{\theta}) \cap (J, \omega') = 0 \in E^{r+s}(A, L).$$

Therefore, the proof is complete. \square

The following is our final result on intersection product.

Theorem 4.5. We use all the notations as in Definition 4.1.

1. With $L = L' = A$, the association

$$((I, \omega), (J, \omega')) \mapsto (I, \omega) \cap (J, \omega') \in E^{r+s}(A, A)$$

defines a bilinear homomorphism

$$\cap : E^r(A, A) \times E^s(A, A) \rightarrow E^{r+s}(A, A).$$

2. With $L' = A$ and $r \geq 2$, the association

$$((I, \omega), (J, \omega')) \mapsto (I, \omega) \cap (J, \omega') \in E^{r+s}(A, L)$$

defines a bilinear homomorphism

$$\cap : E^r(A, L) \times E^s(A, A) \rightarrow E^{r+s}(A, L).$$

3. If $L' \neq A$, $r + s = d$ and $r \geq 2$, the association

$$((I, \omega), (J, \omega')) \mapsto (I, \omega) \cap (J, \omega') \in E^{r+s}(A, LL')$$

defines a bilinear homomorphism

$$\cap : E^r(A, L) \times E^s(A, L') \rightarrow E^d(A, LL').$$

Proof. Proofs of (3), (1) are similar to that of (2). We only prove (2).

Fix $x = (J, \omega') \in G^s(A, A)$ as in Definition 4.1. Suppose I_1 is an ideal of height r and $\omega_1 : \frac{F}{I_1^r} \rightarrow \frac{I_1}{I_1^2}$ is a local orientation. There is an ideal I and ω as in 4.1, such that $(I_1, \omega_1) = (I, \omega) \in E^r(A, L)$. Define

$$\varphi_x(I_1, \omega_1) = (I, \omega) \cap (J, \omega') \in E^{r+s}(A, L).$$

This is well defined by Lemma 4.4. By Lemma 4.2, φ_x factors through a homomorphism

$$\varphi_x : E^r(A, L) \rightarrow E^{r+s}(A, L).$$

Now, the association $x \mapsto \varphi_x$ defines a homomorphism

$$\varphi : G^s(A, A) \rightarrow \text{Hom}(E^r(A, L), E^{r+s}(A, L)).$$

By Lemma 4.2, if x is a global orientation, then $\varphi_x = 0$. So, φ , factors through a homomorphism

$$\cap : E^s(A, A) \rightarrow \text{Hom}(E^r(A, L), E^{r+s}(A, L)).$$

This completes the proof. \square

Forgetting the orientations, we get the following.

Corollary 4.6. There is an intersection product for the weak Euler class groups, corresponding to the same in Theorem 4.5.

Remark 4.7. From the definitions, it follows that the intersection products defined in Theorem 4.5 and Corollary 4.6 are commutative and associative, whenever they are defined.

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