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Excision in algebraic obstruction theory

Satya Mandal [a,*], Yong Yang [b]

[a] *Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd., Lawrence, KS 66045, USA*
[b] *Department of Mathematics and Systems Science, National University of Defense Technology, Changsha, Hunan, 410073, PR China*

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ABSTRACT

In this paper the relative algebraic obstruction groups (also known as Euler class groups) were defined and some excision exact sequences were established. In particular, for a regular domain A, essentially of finite type over an infinite field k, and a rank one projective A-module L_0, it was proved that

$$E^n\ (A[T], L_0 \otimes A[T]) \approx E^n(A, L_0) \quad \text{whenever } 2n \geq \dim A + 4.$$

1. Introduction

In topology, there is a well established obstruction theory for vector bundles (see [11]). A germ for a parallel obstructions theory for projective modules over noetherian commutative rings was given by M.V. Nori around 1990 (see [6,8]). Originally, this program was focused on the top rank case. For projective modules P of rank d, over noetherian commutative rings A with $\dim A = d$, an obstruction class $e(P)$ was defined. Bhatwadekar and Sridharan [3] proved that $e(P) = 0$ if and only if P has a free direct summand. A theory of obstructions for all projective modules, as complete as that of topological vector bundles, is possible. In fact, a parallel K-theoretic approach was initiated by Barge and Morel [1] in 2000. This was given a more complete shape by Fasel [5]. However, this approach does not seem very descriptive and two approaches must reconcile. In this paper, we are concerned with the approach of Nori.

Following [4], obstruction groups $E^n(A, L)$ were defined in [9], for all integers $n \geq 1$ and rank one projective A-modules L. There has been only a limited success in defining obstruction classes $e(P) \in E^n(A, L)$ for projective A-modules P with $rank(P) = n < \dim A$ and $\det(P) \approx L$, as would be desired. When $\det P \approx L \approx A$, we say that P is oriented and the situation is referred to as the oriented case. Otherwise, it is referred to as the non-oriented case. Recently, in the oriented case, Yang [12] defined relative obstructions groups $E^n(A, \mathcal{I}, A)$, with respect to ideals $\mathcal{I}$ of A, when $n \geq 1$. When $2n \geq \dim A + 3$, he established some exact sequences of these groups. The purpose of this paper is to extend the results of Yang [12] to the non-oriented case. As in [12], first we define pull-back homomorphisms $f^* : E^n(A, L) \to E^n(B, B \otimes L)$ of the obstructions groups, corresponding to certain ring homomorphisms $f : A \to B$ and some integers n. Then, we define the relative obstruction groups, $E^n(A, \mathcal{I}, L)$ (see 4.1), and establish some exact sequences in Theorems 4.2 and 4.3. In particular, we establish an excision exact sequence as follows.

Theorem 1.1. *Let A be noetherian commutative ring with $\dim(A) = d$ and $\mathcal{I}$ be an ideal of A. Write $A_0 = \frac{A}{\mathcal{I}}$. Assume that the quotient homomorphism $q : A \twoheadrightarrow A_0$ has a splitting $\beta : A_0 \to A$ such that for each locally n-generated ideal I_0 of A_0, of height n, we have height$(\beta(I_0)A) \geq n$. Suppose L_0 is a projective A_0-module of rank one and $L = L_0 \otimes_\beta A$. Then, for integers n with $2n \geq d + 3$, we have a split exact sequence as follows:*

$$0 \longrightarrow E^n(A, \mathcal{I}, L) \longrightarrow E^n(A, L) \longrightarrow E^n(\tfrac{A}{\mathcal{I}}, L_0) \longrightarrow 0.$$

* Corresponding author.
E-mail addresses: mandal@math.ku.edu (S. Mandal), yangyong@nudt.edu.cn (Y. Yang).

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As an application, we prove that if A is a regular domain, essentially of finite type over an infinite perfect field k, and L_0 is a projective A-module of rank one, then

$$E^n\,(A[T], L_0 \otimes A[T]) \approx E^n(A, L_0) \quad \text{whenever } 2n \geq \dim A + 4.$$

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2. Preliminaries

For the definition of the obstructions groups the readers are referred to [9]. We recall some notations from [9].

Notations 2.1. Throughout this paper, A will denote a commutative noetherian ring with $\dim A = d$ and L will denote a projective A-module of rank one.

(1) For integers $n \geq 1$, write $F = F_n = L \oplus A^{n-1}$.
(2) A local L-orientation (of codimension n) is an equivalence class of surjective homomorphisms $\omega : F/IF \twoheadrightarrow I/I^2$, where I is an ideal of height n. When it is clear from the context, we just call them orientations.
(3) The free abelian group generated by the local orientations (I, ω), where the ideal I is connected, is denoted by $G^n(A, L)$.
(4) The obstruction group of codimension n is defined by $E^n(A, L) := \frac{G^n(A,L)}{\mathcal{R}}$, where $\mathcal{R}$ is the subgroup generated by the global orientations.
(5) A local orientation $\omega : F/IF \twoheadrightarrow I/I^2$, defines an element $(I, \omega) \in G^n(A, L)$. We will take the liberty to use the same notation (I, ω) to denote its image in $E^n(A, L)$.

The following lemma is proved by using standard basic element theory along with generalized dimensions (see [7]).

Lemma 2.2. *Suppose A is a noetherian commutative ring with $\dim A = d$. Suppose I, J are two ideals of A with height$(I) = n$ and $J \subseteq I^2$. Let F be a projective A-module of rank n and $\omega : F \twoheadrightarrow I/J$ be a surjective homomorphism. Also suppose $I_1, \ldots, I_r$ are finitely many ideals of A. Then there is a surjective lift $f : F \twoheadrightarrow I \cap K$, such that (1) $J + K = A$, (2) height$(K) \geq n$ and (3) for $1 \leq i \leq r$, height $\left(\frac{K+I_i}{I_i}\right) \geq n$.*

Proof. Similar to [9, Lemma 4.3]. $\square$

Lemma 2.3. *Suppose A is a noetherian commutative ring with $\dim A = d$. Suppose I, J are two ideals of A and F is a projective A-module of rank n. Let $\omega : F \twoheadrightarrow I/I^2$ and $\varphi : F \twoheadrightarrow \frac{I+J}{J}$ be two surjective homomorphisms such that*

$$\omega \otimes \frac{A}{I+J} = \varphi \otimes \frac{A}{I+J} : \frac{F}{(I+J)F} \twoheadrightarrow \frac{I}{I^2+IJ}.$$

Then, there is a surjective homomorphism $\Omega : F \twoheadrightarrow \frac{I}{I^2 \cap J}$ that lifts both ω and φ.

Proof. Consider the fiber product diagrams:

$$\begin{tikzcd} \frac{A}{I\cap J} \arrow[r] \arrow[d] & \frac{A}{I} \arrow[d] \\ \frac{A}{J} \arrow[r] & \frac{A}{I+J} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \frac{F}{(I\cap J)F} \arrow[rr] \arrow[dd] \arrow[dr, dashed, "\Omega"] & & \frac{F}{IF} \arrow[dd] \arrow[dr, "\omega"] & \\ & \frac{I}{J\cap I^2} \arrow[rr] \arrow[dd] & & \frac{I}{I^2} \arrow[dd] \\ \frac{F}{JF} \arrow[rr] \arrow[dr, "\varphi"] & & \frac{F}{(I+J)F} \arrow[dr] & \\ & \frac{J+I}{J} \arrow[rr] & & \frac{I}{I^2+IJ}. \end{tikzcd}$$

By the properties of fiber product diagrams, the desired homomorphism Ω is defined in the later diagram. This completes the proof. $\square$

Following is the non-oriented version of the theorem of Bhatwadekar and Sridharan [4, Theorem 4.2].

Theorem 2.4. *Let A be a noetherian commutative ring with $\dim A = d$ and $2n \geq d+3$. Let L be a projective A-module of rank one and $F_n = A^{n-1} \oplus L$. Let J be an ideal of height n and $\omega : \frac{F_n}{JF_n} \twoheadrightarrow J/J^2$ be a local L-orientation. Assume*

$$(J, \omega) = 0 \in E^n(A, L).$$

Then there is a surjective lift $\theta : F_n \twoheadrightarrow J$ of ω.

Proof. Similar to that of [4, Theorem 4.2]. □

2.1. Double of a ring

In this subsection, we define the Double of a ring A along an ideal $\mathcal{I}$ and summarize some of the facts about it. Let A be a noetherian commutative ring with $\dim A = d$ and $\mathcal{I}$ be an ideal of A. The **double of A along the ideal $\mathcal{I}$**, is defined as

$$D = D(A, \mathcal{I}) = \{(x, y) \in A \times A : x - y \in \mathcal{I}\}.$$

This will be considered as a subring of $A \times A$. The projection to the first and second coordinates from D will be denoted by p_1 and p_2.

Similarly, for an A-module M, the double of M along the ideal $\mathcal{I}$, is defined as

$$D(M, \mathcal{I}) = \{(m, n) \in M \times M : m - n \in \mathcal{I}M\}.$$

The following are some facts:

(1) The following diagrams

$$\begin{tikzcd} D(A,\mathcal{I}) \arrow[r, "p_1"] \arrow[d, "p_2"'] & A \arrow[d, "q"] \\ A \arrow[r, "q"'] & \frac{A}{\mathcal{I}} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} D(M,\mathcal{I}) \arrow[r, "p_1"] \arrow[d, "p_2"'] & M \arrow[d] \\ M \arrow[r] & \frac{M}{\mathcal{I}M} \end{tikzcd}$$

are fiber product diagrams, where q denotes the quotient homomorphism.

(2) The $kernel(p_1) = 0 \times \mathcal{I}$ and $kernel(p_2) = \mathcal{I} \times 0$.

(3) In fact, the diagonal homomorphism $\Delta : A \to D$ splits both p_1, p_2.

(4) We have $D = \Delta(A) + 0 \times \mathcal{I} = \Delta(A) + \mathcal{I} \times 0$. So, $D = \Delta(A) + \sum \Delta(A)(0, x_i)$, where $\mathcal{I} = \sum Ax_i$. So, D is finitely generated A-module.

(5) So, D is noetherian and integral over A. Therefore, $\dim A = \dim D$.

(6) For any ideal I of A, we have

$$p_1^{-1}(I) = \{(x, x+z) : x \in I, z \in \mathcal{I}\} = \Delta(I) + 0 \times \mathcal{I}$$

and

$$p_2^{-1}(I) = \{(x+z, x) : x \in I, z \in \mathcal{I}\} = \Delta(I) + \mathcal{I} \times 0.$$

(7) If $\wp \in Spec(A)$, then $P = \Delta(\wp) + 0 \times \mathcal{I} = p_1^{-1}(\wp) \in Spec(D)$. Similarly, $P = \Delta(\wp) + \mathcal{I} \times 0 = p_2^{-1}(\wp) \in Spec(D)$.

Lemma 2.5. *Let $A, \mathcal{I}, L, M, D, \Delta$ be as above. Then,*

(1) *There is a natural surjective homomorphism $\tau : D(A, \mathcal{I}) \otimes M \to D(M, \mathcal{I})$, where D is considered as an A-algebra via the diagonal Δ.*

(2) *If Q is projective, then $\tau : D(A, J) \otimes Q \approx D(Q, J)$.*

Proof. It is easy to see that $\tau(m \otimes (x, x+z)) = (xm, (x+z)m)$ is a well defined homomorphism from $D \otimes M \to D(M, J)$. This establishes (1).

If $Q = A^n$ is free, then it is obvious that

$$\tau : D(A, J) \otimes A^n \approx D(A^n, J).$$

In the general case, note that there is a split exact sequence:

$$0 \longrightarrow Q \xrightarrow{g} F \xrightarrow{f} P \longrightarrow 0$$

where F is free. Correspondingly, we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & D(A,J)\otimes Q \arrow[r] \arrow[d, "\tau"] & D(A,J)\otimes F \arrow[r] \arrow[d, "\wr"] & D(A,J)\otimes P \arrow[r] \arrow[d, "\tau"] & 0 \\
0 \arrow[r] & D(Q,J) \arrow[r, "\gamma"'] & D(F,J) \arrow[r, "\varphi"'] & D(P,J) \arrow[r] & 0.
\end{tikzcd}$$

Here the rows are exact, while one needs a proof that the bottom row is exact. It is easy to see that γ is injective and φ is surjective and $\varphi\gamma = 0$. Suppose $\varphi(m, m+z) = 0$ for some $m \in F$, $z \in \mathcal{I}F$. Then $f(m) = f(z) = 0$. Therefore, $g(u) = m$ and $g(v) = z$ for some $u, v \in Q$. Let $\epsilon : F \to Q$ be a splitting of g. Then $v = \epsilon g(v) = \epsilon(z) \in \mathcal{I}Q$. Hence $(u, u+v) \in D(Q, \mathcal{I})$ and $\gamma(u, u+v) = (m, m+z)$. This establishes that the bottom row is exact.

Since the middle vertical map is an isomorphism, the first vertical map is injective. This completes the proof. (*Alternately, one could use the Snake lemma to prove the same.*) $\square$

The following lemma will be of our interest subsequently.

Lemma 2.6. *With the notations as above, let J be an ideal of the double $D = D(A, \mathcal{I})$. If $height(J) = n$, then $height(p_1(J)) \geq n$.*

Proof. Suppose $p_1(J) \subseteq \wp \in Spec(R)$ are minimal. We will prove that $height(\wp) \geq n$. We have $J \subseteq p_1^{-1}(p_1(J)) \subseteq p_1^{-1}(\wp)$. There is a prime ideal $P \in Spec(D)$ such that P is minimal over J and $P \subseteq p_1^{-1}(\wp)$. Write $m = height(P)$. Then $m \geq n = height(J)$. Let

$$P_0 \subseteq P_1 \subseteq P_2 \cdots \subseteq P_m = P$$

be a strictly increasing chain of primes in $Spec(D)$. Write, $\wp_i = P_i \cap R = \Delta^{-1}(P_i)$. Since, $R \to D$ is integral, there is no inclusion relationship between two primes in D over the same prime $\wp \in Spec(R)$ (see [10, Theorem 9.3, pp. 66]). So,

$$\wp_0 \subseteq \wp_1 \subseteq \wp_2 \cdots \subseteq \wp_m$$

is a strictly increasing chain in $Spec(R)$. Further, $\wp_m \subseteq p_1(P) \subseteq p_1(p_1^{-1}(\wp)) = \wp$. Therefore, $height(\wp) \geq m \geq n$. The proof is complete. $\square$

3. Pull-back and functoriality

In this section, we define some pull-back homomorphisms of the obstruction groups, corresponding to some suitable ring homomorphisms. First one will correspond to the quotient homomorphisms $q : A \to A/J$.

Definition 3.1. Let A be a noetherian commutative ring with $\dim A = d$ and $J \subseteq A$ be an ideal. Let L be a rank one projective A-module. For integers n, with $2n \geq d+3$, there is a group homomorphism

$$\rho = \rho_J : E^n(A, L) \to E^n\left(\frac{A}{J}, \frac{L}{JL}\right)$$

defined as follows:
Write $F = L \oplus A^{n-1}$. Let $\omega : F/IF \twoheadrightarrow I/I^2$ be local L-orientation. We can find an ideal I_1 and a local orientation $\omega_1 : F/I_1F \twoheadrightarrow I_1/I_1^2$ such that $(I, \omega) = (I_1, \omega_1)$ and $height\left(\frac{I_1+J}{J}\right) \geq n$. Then, ω_1 induces an orientation β as in the following commutative diagram:

$$\begin{tikzcd}
F/I_1F \arrow[r, two heads, "\omega_1"] \arrow[d, two heads] & I_1/I_1^2 \arrow[d, two heads] \\
\frac{F}{(I_1+J)F} \arrow[r, two heads, "\beta"'] & \frac{I_1+J}{I_1^2+J}.
\end{tikzcd}$$

Define

$$\rho(I, \omega) = \left(\frac{I_1 + J}{J}, \beta\right).$$

We use the notations $q^* = E^n(q) = \rho = \rho_J$, corresponding to notation for the quotient map $q : A \to A/J$. This homomorphism will be called a **pull-back** homomorphism.

Proof that ρ Well Defined. First, we define a homomorphism

$$\varphi : G^n(A, L) \to E^n\left(\frac{A}{J}, \frac{L}{JL}\right) \quad \text{by} \quad \varphi(I, \omega) = \left(\frac{I_1 + J}{J}, \beta\right) \in E^n\left(\frac{A}{J}, \frac{L}{JL}\right),$$

where β is as above. Two different representatives of ω_1 leads to the same, because transvections of F/I_1F will induce transvections of $F_1/(J + I_1)F$.

First, we prove that if (I_1, ω_1) is global, then so is the image. If it is global, then ω_1 lifts to a surjective homomorphism $f : F \twoheadrightarrow I_1$. Then, f induces a surjective lift g of β, as demonstrated by the following diagram:

$$\begin{tikzcd}
F \arrow[rr, two heads, "f"] \arrow[dr] \arrow[dd] & & I_1 \arrow[dr] \arrow[dd] & \\
& F/JF \arrow[rr, "g"] \arrow[dd] & & \frac{I_1+J}{J} \arrow[dd] \\
F/I_1F \arrow[rr, two heads, "\omega_1"] \arrow[dr] & & I_1/I_1^2 \arrow[dr] & \\
& \frac{F}{(I_1+J)F} \arrow[rr, "\beta"] & & \frac{I_1+J}{I_1^2+J}.
\end{tikzcd}$$

This establishes that β is global. Now, suppose

$$(I, \omega) = (I_1, \omega_1) = (I_2, \omega_2) \in E^n(A, L), \quad \text{with } height(I_1/J) = height(I_2/J) = n.$$

There is an ideal K and a surjective lift $f : F \twoheadrightarrow I_1 \cap K$ of ω_1, such that $I_1 + K = A$, $height(K) \geq n$. Since $2n > d$, we can also assume $K + I_2 = A$. Let $\omega_K : F/KF \twoheadrightarrow K/K^2$ be induced by f. We have,

$$(I_1, \omega_1) + (K, \omega_K) = (I_2, \omega_2) + (K, \omega_K) = 0.$$

By Theorem 2.4, there is a surjective homomorphism $f_2 : F \twoheadrightarrow I_2 \cap K$ that lifts ω_2 and ω_K. So, both $(I_1 \cap K, \omega_1 \otimes \omega_K)$, $(I_2 \cap K, \omega_2 \otimes \omega_K)$ are global. Since the image of a global orientation is global, the images of both (I_1, ω_1), (I_2, ω_2) are negative of that of (K, ω_K).

Therefore, the homomorphism φ is well defined. Again, since image of a global orientation is global, φ factors through a homomorphism

$$\rho : E^n(A, L) \to E^n\left(\frac{A}{J}, \frac{L}{JL}\right).$$

This completes the proof that ρ is well defined. $\square$

Proposition 3.2. *Let A, J, d, n, L be as in Definition 3.1. Let J_1 be another ideal. Then the diagram*

$$\begin{tikzcd}
E^n(A, L) \arrow[r, "\rho_J"] \arrow[dr, "\rho_{J+J_1}"'] & E^n\left(\frac{A}{J}, \frac{L}{JL}\right) \arrow[d, "\rho_{(J_1+J)/J}"] \\
& E^n\left(\frac{A}{J+J_1}, \frac{L}{(J+J_1)L}\right)
\end{tikzcd}$$

commutes.

Proof. Write $\rho_0 = \rho_{J+J_1}$, $\rho = \rho_J$, $\rho_1 = \rho_{A/(J+J_1)}$. Write $\mathcal{J} = J + J_1$ and $F = L \oplus A^{n-1}$. Let $\omega : F/IF \twoheadrightarrow I/I^2$ be a local L-orientation. We can find I_1 and a local L-orientation $\omega_1 : F/I_1F \twoheadrightarrow I_1/I_1^2$ such that (1) $(I, \omega) = (I_1, \omega_1)$, (2) $height\left(\frac{I_1+J}{J}\right) \geq n$ and (3) $height\left(\frac{I_1+J+J_1}{J+J_1}\right) = height\left(\frac{I_1+\mathcal{J}}{\mathcal{J}}\right) \geq n$. We have,

$$\rho_1\rho(I, \omega) = \rho_1\left(\frac{I_1 + J}{J}, \beta\right) = \left(\frac{I_1 + J + J_1}{J + J_1}, \gamma\right) = \rho_0(I, \omega)$$

where β, γ are induced by ω_1, according to the following commutative diagram:

$$\begin{tikzcd} F/I_1F \arrow[r, two heads] \arrow[d, "\omega_1"'] & \frac{F}{(I_1+J)F} \arrow[r] \arrow[d, "\beta"'] & \frac{F}{(I_1+J+J_1)F} \arrow[d, "\gamma"] \\ I_1/I_1^2 \arrow[r] & \frac{I_1+J}{I_1^2+J} \arrow[r] & \frac{I_1+J+J_1}{I_1^2+J+J_1}. \end{tikzcd}$$

This completes the proof. □

Next, we define pull-back homomorphisms of the obstruction groups, corresponding to some suitable ring homomorphisms $R \to A$.

Definition 3.3. Suppose $f : R \to A$ is a homomorphism of two noetherian commutative rings, with $\dim R = d_1$, $\dim A = d_2$. Let $n \geq 1$ be an integer. For an ideal I of R, we will denote $IA := f(I)A$. Assume that for any ideal I of R, which is locally generated by n elements and $height(I) = n$, we have $height(IA) \geq n$.

Let L be a projective R-module of rank one and $L' = L \otimes A$. Then, there is a homomorphism

$$f^* = E(f) = E^n(f) : E^n(R, L) \to E^n(A, L')$$

defined below. This homomorphism will be called **a pull-back homomorphism**.

To define $E^n(f)$, we proceed as follows. Write $F = F_n = L \oplus R^{n-1}$, $F' = F \otimes A$. Let I be an ideal of R of height n and $\omega : F/IF \twoheadrightarrow I/I^2$ be a local L-orientation. Write $J = IA$. Then ω induces a local L'-orientation ω' by the following commutative diagram:

$$\begin{tikzcd} F/IF \arrow[r, "\omega"] \arrow[d] & I/I^2 \arrow[d] \\ \frac{F\otimes A}{I(F\otimes A)} \arrow[r, "\omega'"'] & \frac{IA}{I^2A}. \end{tikzcd}$$

The association $(I, \omega) \mapsto (J, \omega') \in E^n(A, L')$ defines a group homomorphism

$$\varphi_0 : G^n(R, L) \to E^n(A, L').$$

If (I, ω) is global, ω lifts to a surjection $\Omega : F \twoheadrightarrow I$. It is easy to see that $\Omega \otimes Id_A$ induces a surjective lift $\Omega' : F \otimes A \twoheadrightarrow J$ of ω'. So, $\varphi_0((I, \omega)) = 0$. Therefore, φ_0 factors through a homomorphism

$$E^n(f) : E^n(R, L) \to E^n(A, L').$$

This completes the definition of $E^n(f)$ and establishes that it is well defined. □

Proposition 3.4. *Let $f : R \to A$, $g : A \to B$ be homomorphisms of noetherian commutative rings of finite dimension, satisfying the properties of Definition 3.3. Let L be a rank one projective R-module. Then the diagram*

$$\begin{tikzcd} E^n(R, L) \arrow[r, "E(f)"] \arrow[dr, "E(gf)"'] & E^n(A, L\otimes A) \arrow[d, "E(g)"] \\ & E^n(B, L\otimes B) \end{tikzcd}$$

commutes.

Proof. Obviously from the Definition 3.3. □

Proposition 3.5. *Let $f : A \to R$ be a homomorphism of commutative noetherian rings of finite dimension, and L be a rank one projective A-module. Let $r, s \geq 1$ be two integers be such that $r \geq 2$, if $L \neq A$. Assume that f satisfies the properties of Definition 3.3 for $n = r, s, r+s$. Then, for $x \in E^r(A, L)$, $y \in E^s(A, A)$, we have*

$$E(f)(x) \cap E(f)(y) = E(f)(x \cap y) \in E^{r+s}(A, L \otimes A),$$

where $\cap$ is defined as in [9, Theorem 4.5].

Proof. Write $F = L \oplus A^{r-1}$, $F' = A^s$. Using bilinearity, we can assume that $x = (I, \omega)$ and $y = (J, \omega')$, where $I, J \subseteq A$ are ideals with $height(I) = r$, $height(J) = s$ and $\omega : F/IF \twoheadrightarrow I/I^2$, $\omega' : F'/JF' \twoheadrightarrow J/J^2$ are local orientations. We can assume that $height(I + J) = r + s$. So,

$$x \cap y = (I + J, \omega_0) \quad \text{where } \omega_0 : \frac{F \oplus F'}{(I + J)F \oplus F'} \twoheadrightarrow \frac{(I + J)}{(I + J)^2}$$

is induced by ω, ω'.

Since $f : A \to R$ satisfies properties of Definition 3.3 for $n = r, s, r+s$ the proposition follows by chasing the definitions. $\Box$

Remark 3.6. The referee points out that the Definitions 3.1 and 3.3 can be combined to define pull-back homomorphisms for more general ring homomorphisms $f : A \to B$. For example, assume that both A, B contain a field k and f is a k-algebra homomorphism. Then, the projection homomorphism $p : A \to A \otimes_k B$ is flat, since B is flat over k and it satisfies the conditions of Definition 3.3. So, for integers $n \geq 1$ and any rank one projective A-module L, the pull-back $p^* : E^n(A, L) \to E^n(A \otimes_k B, L \otimes_k B)$ is defined. Now consider the surjective homomorphism $q : A \otimes_k B \to B$ given by $q(x \otimes y) = f(x)y$. Then $f = qp$. If $2n \geq \dim(A \otimes B) + 3$ then the pull-back $q^* : E^n(A \otimes B, L \otimes_k B) \to E^n(B, L \otimes_A B)$ is defined. So, the pull-back $f^* := q^*p^* : E^n(A, L) \to E^n(B, L \otimes_A B)$ is defined. (*In fact, the argument works, if k is any commutative ring and $k \to B$ is flat.*) The authors are thankful to the referee for this comment.

4. Exact sequences and excision

In this section, first we define the relative obstruction groups and then establish the exact sequences.

Definition 4.1. Let A be a commutative noetherian ring and $\mathcal{I}$ be an ideal of A. Let L be a projective A-module with $rank(L) = 1$. Let $n \geq 1$ be an integer. With $D = D(A, \mathcal{I})$ and other notations as in Section 2.1, by Lemma 2.6 the pull-back homomorphism

$$E(p_1) : E^n(D, D \otimes L) \to E^n(A, L)$$

is well defined. The relative obstruction groups, $E^n(A, \mathcal{I}, L)$ are defined as

$$E^n(A, \mathcal{I}, L) = Kernel\,(E(p_1))\,.$$

The following theorem establishes an exact sequence.

Theorem 4.2. *Let A be a noetherian commutative ring with $\dim(A) = d$ and $\mathcal{I}$ be an ideal. As before, $p_1, p_2 : D(A, \mathcal{I}) \to A$ will denote the projections to the first and second coordinates. Suppose L is a projective A-module of rank one. For integers n with $2n \geq d+3$, the following*

$$E^n(A, \mathcal{I}, L) \xrightarrow{E(p_2)} E^n(A, L) \xrightarrow{\rho_{\mathcal{I}}} E^n\left(\frac{A}{\mathcal{I}}, \frac{L}{\mathcal{I}L}\right)$$

is an exact sequence.

Proof. Let $q : A \to A/\mathcal{I}$ denote the quotient homomorphism and $F = L \oplus A^{n-1}$. First, we prove $\rho E(p_2) = 0$. Suppose

$$(W, \omega) \in E^n(A, \mathcal{I}, L) \subseteq E^n(D, D \otimes L) \quad \text{where } \omega : \frac{D \otimes F}{W(D \otimes F)} \twoheadrightarrow W/W^2$$

is a local orientation, with ideal $W \subseteq D$ of height n. By Lemma 2.2, we can assume that *height* $\left(\frac{W + 0 \times \mathcal{I}}{0 \times \mathcal{I}}\right) \geq n$ and *height* $\left(\frac{W + \mathcal{I} \times 0}{\mathcal{I} \times 0}\right) \geq n$. Since $p_1 : \frac{D}{0 \times \mathcal{I}} \xrightarrow{\sim} \frac{A}{\mathcal{I}}$,

$$height\left(\frac{p_1(W) + \mathcal{I}}{\mathcal{I}}\right) = height\left(\frac{W + 0 \times \mathcal{I}}{0 \times \mathcal{I}}\right) \geq n.$$

Write $J_1 = p_1(W)$, and $J_2 = p_2(W)$. Since, $qp_1 = qp_2$, we have, $J_1 + \mathcal{I} = J_2 + \mathcal{I}$. For $i = 1, 2$ ω induces ω_i, β_i by the commutative diagram:

$$\begin{tikzcd}
\frac{D \otimes F}{W(D \otimes F)} \arrow[r, "p_i"] \arrow[d, "\omega"'] & \frac{F}{J_i F} \arrow[r] \arrow[d, "\omega_i"'] & \frac{F}{(J_i + \mathcal{I})F} \arrow[d, "\beta_i"'] \\
W/W^2 \arrow[r, "p_i"'] & J_i/J_i^2 \arrow[r] & \frac{J_i + \mathcal{I}}{J_i^2 + \mathcal{I}}.
\end{tikzcd}$$

Then, for $i = 1, 2$ we have $E(p_i)(W, \omega) = (J_i, \omega_i) \in E^n(A, L)$. It follows, $\beta_1 = \beta_2$. Since height is consistent,

$$\rho_{\mathcal{I}} E(p_2)(W, \omega) = \left(\frac{J_2 + \mathcal{I}}{\mathcal{I}}, \beta_2\right) = \left(\frac{J_1 + \mathcal{I}}{\mathcal{I}}, \beta_1\right) = \rho_{\mathcal{I}} E(p_1)(W, \omega) = 0.$$

Conversely, suppose $x \in E^n(A, L)$, are such that $\rho_{\mathcal{I}}(x) = 0$. Since, $2n > d$, we can write $x = (I, \omega)$ where $I \subseteq A$ is an ideal of height n and $\omega : F/IF \twoheadrightarrow I/I^2$ is a local orientation. We can further assume that *height* $\left(\frac{I + \mathcal{I}}{\mathcal{I}}\right) \geq n$. Now, ω induces

ω' as in the following commutative diagram:

$$\begin{tikzcd}
F/IF \arrow[r, two heads, "\omega"] \arrow[d] & I/I^2 \arrow[d] \\
\frac{F}{(I+\mathcal{J})F} \arrow[r, two heads, "\omega'"'] & \frac{I+\mathcal{J}}{I^2+\mathcal{J}}.
\end{tikzcd}$$

Since $\rho(x) = \left(\frac{I+\mathcal{J}}{\mathcal{J}}, \omega'\right) = 0$, we have ω' lifts to a surjection $f' : \frac{F}{\mathcal{J}F} \twoheadrightarrow \frac{I+\mathcal{J}}{\mathcal{J}}$. We have the following fiber product diagram:

$$\begin{tikzcd}
F/(I \cap \mathcal{J})F \arrow[rr] \arrow[dr, dashed, two heads, "f"] \arrow[dd] & & F/IF \arrow[dr, two heads, "\omega"] \arrow[dd] & \\
& I/\mathcal{J} \cap I^2 \arrow[rr] \arrow[dd] & & I/I^2 \arrow[dd] \\
F/\mathcal{J}F \arrow[rr] \arrow[dr, two heads, "f'"] & & F/(I+\mathcal{J})F \arrow[dr, two heads, "\omega'"] & \\
& (\mathcal{J}+I)/\mathcal{J} \arrow[rr] & & (I+\mathcal{J})/(I^2+\mathcal{J})
\end{tikzcd}$$

where f is defined by properties of fiber product diagrams. By Lemma 2.3, there is a surjective lift $\varphi : F \twoheadrightarrow I \cap K$, of f, such that (1) $K + \mathcal{J} \cap I^2 = A$, (2) *height* $(K) \geq n$ and (3) *height* $\left(\frac{K+\mathcal{J}}{\mathcal{J}}\right) \geq n$.

Now, φ defines an element $(K, \omega_K) = -(I, \omega) \in E^n(A, L)$. Define, $W = \Delta(K) + \mathcal{J} \times 0 = \{(x, y) \in D : y \in K\}$. In fact, W is defined by the fiber product of A and K as follows:

$$\begin{tikzcd}
W \subseteq D \arrow[r, "p_1"] \arrow[d, "p_2"'] & A = K + \mathcal{J} \arrow[d] \\
K \arrow[r] & \frac{K+\mathcal{J}}{\mathcal{J}} = A/\mathcal{J}.
\end{tikzcd}$$

By Lemma 2.5, $D \otimes F = D(F, J)$. We consider the following fiber product diagram:

$$\begin{tikzcd}
D \otimes F \arrow[rr, "p_1"] \arrow[dr, dashed, two heads, "\omega_W"] \arrow[dd, "p_2"'] & & F \arrow[dr, two heads, "0"] \arrow[dd] & \\
& W/W^2 \arrow[rr, "p_1"] \arrow[dd] & & 0 \arrow[dd] \\
F \arrow[rr] \arrow[dr, two heads, "\omega_K"'] & & F/\mathcal{J}F \arrow[dr, two heads, "0"] & \\
& K/K^2 \arrow[rr] & & 0.
\end{tikzcd}$$

Here ω_K is defined by properties of fiber product diagrams. (*Alternately, one can prove elementwise that* $W/W^2 \to K/K^2$ *is an isomorphism, using the fact that* $x + y = 1$ *for some* $x \in K^2$, $y \in \mathcal{J}^2$ *and* $(1, x) = (x, x) + (y, 0) \in W^2$.)

We consider ω_W as an orientation. It follows $E(p_2)(W, \omega_W) = (K, \omega_K) = -(I, \omega) = -x$. So, $x \in$ *image*$E(p_2)$. Since $p_1(W) = A$, we have $(W, \omega_W) \in$ *ker*$(E(p_1)) = E^n(A, L, \mathcal{J})$. This completes the proof of (4.2). □

With further conditions, we extend the above sequences as follows.

Theorem 4.3. *Use notations as in Theorem 4.2 and assume* $2n \geq d+3$. *Write* $A_0 = \frac{A}{\mathcal{J}}$. *Assume that the quotient homomorphism* $q : A \twoheadrightarrow A_0$ *has a splitting* $\beta : A_0 \to A$ *and* $L = L_0 \otimes_\beta A_0$ *for some rank one projective* A_0*-module* L_0.

(1) *Then, the sequence*

$$0 \longrightarrow E^n(A, \mathcal{J}, L) \xrightarrow{E(p_2)} E^n(A, L) \xrightarrow{\rho_{\mathcal{J}}} E^n\left(\frac{A}{\mathcal{J}}, \frac{L}{\mathcal{J}L}\right)$$

is exact.

(2) *Assume that, for each locally n-generated ideal I_0 of A_0, of height n, we have height$(\beta(I_0)A) \geq n$. Then, the sequence*

$$E^n(A, \mathcal{I}, L) \xrightarrow{E(p_2)} E^n(A, L) \xrightarrow{\rho_{\mathcal{I}}} E^n\left(\tfrac{A}{\mathcal{I}}, \tfrac{L}{\mathcal{I}L}\right) \longrightarrow 0$$

is exact and $\rho_{\mathcal{I}}$ splits.

Proof. First, we prove (2). Because of Theorem 4.2, we only need to prove that ρ is split-surjective. Since $E(\beta)$ is defined, it is easy to check that $\rho_{\mathcal{I}}E(\beta) = Id$. So, $\rho_{\mathcal{I}}$ is surjective and splits.

Again by Theorem 4.2, to prove (1), we need to prove that $E(p_2)$ is injective on $E^n(A, \mathcal{I}, L)$. Let $x \in E^n(A, \mathcal{I}, L) \subseteq E^n(D, L \otimes D)$ such that $E(p_2)(x) = 0$. Since $2n > d = \dim(D)$, we can assume $x = (W, \omega)$, where $F = L \oplus A^{n-1}$ and W is an ideal of D with *height*$(W) \geq n$ and $\omega : \frac{F\otimes D}{W(F\otimes D)} \twoheadrightarrow \frac{W}{W^2}$ is a local orientation.

Since $L = L_0 \otimes_\beta A$, we have $L_0 = L \otimes_A A_0$. We will use the notation $F' = F \otimes D$. Consider the following fiber product diagram:

$$\begin{tikzcd}
\frac{F'}{(W\cap(\mathcal{I}\times\mathcal{I}))F'} \arrow[rr] \arrow[dr, dashed, "f"] \arrow[dd] & & \frac{F'}{WF'} \arrow[dr, "\omega"] \arrow[dd] & \\
& \frac{W}{W^2\cap(\mathcal{I}\times\mathcal{I})} \arrow[rr] \arrow[dd] & & \frac{W}{W^2} \arrow[dd] \\
\frac{F'}{(\mathcal{I}\times\mathcal{I})F'} \arrow[rr] \arrow[dr, "f'"] & & \frac{F'}{(W+(\mathcal{I}\times\mathcal{I}))F'} \arrow[dr, "\omega\otimes\frac{D}{\mathcal{I}\times\mathcal{I}}"] & \\
& \frac{W+(\mathcal{I}\times\mathcal{I})}{(\mathcal{I}\times\mathcal{I})} \arrow[rr] & & \frac{W+(\mathcal{I}\times\mathcal{I})}{W^2+(\mathcal{I}\times\mathcal{I})}.
\end{tikzcd}$$

In this diagram, f' is a surjective lift of $\omega \otimes \frac{D}{\mathcal{I}\times\mathcal{I}}$, which exists, by Theorem 2.4, because $\left(\frac{W+\mathcal{I}\times\mathcal{I}}{\mathcal{I}\times\mathcal{I}}, \omega\otimes\frac{D}{\mathcal{I}\times\mathcal{I}}\right) = \rho E(p_1)(W, \omega) = 0$. Also, f is given by the properties of fiber product diagrams and is surjective.

By Lemma 2.2, f lifts to a surjection, $\Omega : F' \twoheadrightarrow W \cap K$ where K is an ideal with (1) $W + K = D$, (2) *height*$(K) \geq n$, (3) *height* $\left(\frac{K+0\times\mathcal{I}}{0\times\mathcal{I}}\right) \geq n$ (4) *height* $\left(\frac{K+\mathcal{I}\times 0}{\mathcal{I}\times 0}\right) \geq n$ (5) $K + \mathcal{I} \times \mathcal{I} = D$.

Ω induces a local orientation $\omega_K : F'/KF' \twoheadrightarrow K/K^2$. We have $(I, \omega) + (K, \omega') = 0$. Write $K_1 = p_1(K), K_2 = p_2(K_2)$. Note $q(K_1) = q(K_2) = image(K) = A_0$, and *height*$(K_i) \geq n$ for $i = 1, 2$. For $i = 1, 2$, let $\omega_i : F/K_iF \twoheadrightarrow K_i/K_i^2$ be induced by ω_K.

The following are some observations:

(1) Claim: $p_1^{-1}(K_1) = K + 0 \otimes \mathcal{I}$ and $p_2^{-1}(K_2) = K + \mathcal{I} \otimes 0$.
To see this, note $K + 0 \otimes \mathcal{I} \subseteq p_1^{-1}(K)$. Conversely, let $(x, y) \in p^{-1}(K)$. Then, $(x, y') \in K$ for some $y' \in A$. Therefore, $(x, y) = (x, y') + (0, (y - y')) = (x, y') + (0, (y - x) + (x - y')) \in K + 0 \otimes \mathcal{I}$. This establishes the claim.
(2) Since $K + \mathcal{I} \times \mathcal{I} = D$ we have $K_i + \mathcal{I} = A$ for $i = 1, 2$.
(3) Also $E(p_i)(K, \omega_K) = (K_i, \omega_i) = 0$ for $i = 1, 2$. Therefore, by Theorem 2.4, there are surjective lifts $\Omega_i : F \twoheadrightarrow K_i$ of ω_i.

Write $F_0 = L_0 \oplus A_0^{n-1}$. Then, $F \otimes A_0 = L \otimes A_0 \oplus A_0^{n-1} \approx F_0$. Since $K_i + \mathcal{I} = A$, for $i = 1, 2$, we have, $\Omega_i \otimes A_0$ surjects on to A_0. Write $\Omega_i^0 = \Omega_i \otimes_A A_0 : F_0 \twoheadrightarrow A_0$ for $i = 1, 2$.

Write $J_1 = \{(a, a + z) : a \in K_1, z \in \mathcal{I}\}$ and $J_2 = \{(b + z, b) : b \in K_2, z \in \mathcal{I}\}$. Both J_1 and J_2 are ideals of D. Consider the following two fiber product diagrams:

$$\begin{tikzcd}
F' \arrow[rr, "p_1"] \arrow[dr, dashed, "\Gamma_1"] \arrow[dd, "p_2"'] & & F \arrow[dr, "\Omega_1"] \arrow[dd] & \\
& J_1 \arrow[rr] \arrow[dd] & & K_1 \arrow[dd] \\
F \arrow[rr] \arrow[dr, "\Omega_1^0\otimes_\beta A"'] & & F_0 \arrow[dr, "\Omega_1^0"] & \\
& A \arrow[rr] & & A_0
\end{tikzcd}
\qquad
\begin{tikzcd}
F' \arrow[rr, "p_1"] \arrow[dr, dashed, "\Gamma_2"] \arrow[dd, "p_2"'] & & F \arrow[dr, "\Omega_2^0\otimes_\beta A"] \arrow[dd] & \\
& J_2 \arrow[rr] \arrow[dd] & & A \arrow[dd] \\
F \arrow[rr] \arrow[dr, "\Omega_2"'] & & F_0 \arrow[dr, "\Omega_2^0"] & \\
& K_2 \arrow[rr] & & A_0.
\end{tikzcd}$$

Here Γ_1, Γ_2 are obtained by the properties of fiber product diagrams and they are surjective by the same. We gather some facts below:

(1) We have $J_1 + J_2 = D$.

Proof. We have $K + \mathcal{I} \times \mathcal{I} = D$. So, $(x, y) + (u, v) = (1, 1)$ for some $(x, y) \in K$ and $u, v \in \mathcal{I}$. So, $(1, y) = (y + v, y) \in J_2$ and $(x, 1) = (x, x + u) \in J_1$. Therefore, $(1, 1) = (1, y) + (x, 1)(0, v) \in J_1 + J_2$. This establishes $J_1 + J_2 = D$ or (1). $\square$

(2) We have $K = J_1 \cap J_2$.

Proof. Clearly, for $(a, b) \in K$, we have $(a, b) = (a, a + (b - a)) \in J_1$. Similarly, $(a, b) \in J_2$. Therefore, $K \subseteq J_1 \cap J_2$.

Now, let $(a, b) \in J_1 \cap J_2$. Looking at the description of J_i, it follows $a \in K_1$ and $b \in K_2$. So, $(a, c) \in K$ for some $c \in A$. So, $(a, b) = (a, c) + (0, b - c)$. Since $(a, c) \in K \subseteq J_1 \cap J_2$, we have $(0, b - c) \in J_1 \cap J_2$. Therefore, $b - c \in K_2 \cap \mathcal{I} = K_2\mathcal{I}$. So, $b - c = \sum y_i z_i$ for some $(x_i, y_i) \in K$ and $z_i \in \mathcal{I}$. Hence

$$(a, b) = (a, c) + (0, b - c) = (a, c) + \sum (x_i, y_i)(0, z_i) \in K.$$

This establishes (2). $\square$

(3) Since $K = J_1 \cap J_2$, we have *height* $(J_i) \geq n$. So, Γ_i induce local orientation $\gamma_i : F'/J_iF' \twoheadrightarrow J_i/J_i^2$ for $i = 1, 2$. Indeed, they are global. So,

$$(J_1, \gamma_1) = (J_2, \gamma_2) = 0 \in E^n(D, L \otimes D).$$

(4) In fact, $\omega_K \otimes D/J_i = \gamma_i$ for $i = 1, 2$.

Proof. We will check for $i = 1$. We will check that $\Omega(m) - \Gamma_1(m) \in J_1^2$, for $m \in F'$. Since $K_1^2 + \mathcal{I}^2 = A$, we have $x + c = 1$ for some $(x, y) \in K^2$ and $c \in \mathcal{I}^2$. So, $(x, 1) = (x, y) + (0, (1 - x) + (x - y)) \in J_1^2$. Therefore, for any $z \in \mathcal{I}$ we have $(0, z) = (x, 1)(0, z) \in J_1^2$.

Let $m \in F'$ and $\Gamma_1(m) = (a, b)$, $\Omega(m) = (x, y)$. Since the first coordinates of both agree in K_1/K_1^2, we have $c = x - a \in K_1^2$. Therefore, $(c, d) \in K^2 \subseteq J_1^2$ for some $d \in A$. So,

$$(x, y) - (a, b) = (c, y - b) = (c, d) + (0, y - b - d) \in J_1^2.$$

This establishes (4). $\square$

It follows from above,

$$(K, \omega_K) = (J_1, \gamma_1) + (J_2, \gamma_2) = 0.$$

Hence $(W, \omega) = -(K, \omega_K) = 0$. The proof is complete. $\square$

As in the paper [12] of Yang, immediate application of Theorem 4.3 would provide exact sequences for Euler class groups of polynomial rings and Laurent polynomial rings.

Corollary 4.4. *Let R be a commutative ring with $\dim R = d$. Let $A = R[X]$ be the polynomial ring and $B = R[X, X^{-1}]$ be the Laurent polynomial ring. Let L_0 be a projective R-module of rank one. Write $L = L_0 \otimes A$, $L' = L_0 \otimes B$. Assume that $2n \geq d + 4$. We have the following.*

(1) *The sequence,*

$$0 \longrightarrow E^n(A, (X), L) \xrightarrow{E(p_2)} E^n(A, L) \xrightarrow{\rho_X} E^n(R, L_0) \longrightarrow 0$$

is a split exact sequence.

(2) *The sequence,*

$$0 \longrightarrow E^n\left(B, (X - 1), L'\right) \xrightarrow{E(p_2)} E^n\left(B, L'\right) \xrightarrow{\rho_{X-1}} E^n(R, L_0) \longrightarrow 0$$

is a split exact sequence.

(3) *Further, if R is a regular domain that is essentially of the finite type over an infinite field k, then*

$$\rho_X : E^n(A, L) \xrightarrow{\sim} E^n(R, L_0)$$

is an isomorphism. In particular, the relative group

$$E^n(A, (X), L) = 0.$$

Proof. Obviously, (1) and (2) are direct consequences of Theorem 4.3. To prove (3), we need to show that the first homomorphism is zero. As in Theorem 4.3, p_1, p_2 will denote the projection maps and $q : A \twoheadrightarrow R$ will be the quotient homomorphism. The first homomorphism in the sequence is $E(p_2)$. Suppose $x \in E^n(A, (X), L)$. We will prove $E(p_2)(x) = 0$. We can assume that $x = (I, \omega)$ where I is an ideal of $D = D(A, (X))$, with *height* $(I) \geq n$, *height* $(p_1(I)) \geq n$ and *height* $(p_2(I)) \geq n$. Write $p_1(I) = I_1$, $p_2(I) = I_2$ and $I_0 = q(I_1) = q(I_2)$. Therefore, $I_0 = (I_1, X) = (I_2, X)$.

For $i = 0, 1, 2$, let $\omega_i : F/I_iF \twoheadrightarrow I_i/I_i^2$ be induced by ω. From the exactness of the sequence, we have

$$(I_0, \omega_0) = \rho_X(I_2, \omega_2) = \rho_X E(p_2)(I, \omega) = 0.$$

Therefore, by Theorem 2.4, ω_0 lifts to a surjective homomorphism $f_0 : L_0 \oplus R^{n-1} \twoheadrightarrow I_0$. Now, by [2, Theorem 4.13], there is a surjective lift $f_2 : L \oplus A^{n-1} \twoheadrightarrow I_2$ such that $f_2 \otimes A/(X) = f_0$. So, $(I_2, \omega_2) = 0$. This completes the proof. $\square$

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