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On the complete intersection conjecture of Murthy

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ABSTRACT

Suppose $A = k[X_1, X_2, \ldots, X_n]$ is a polynomial ring over a field k and I is an ideal in A. M.P. Murthy conjectured that $\mu(I) = \mu(I/I^2)$, where μ denotes the minimal number of generators. Recently, Fasel [3] settled this conjecture, affirmatively, when k is an infinite perfect field, with $1/2 \in k$ (always). We are able to do the same, when k is an infinite field. In fact, we prove similar results for ideals I in a polynomial ring $A = R[X]$, that contains a monic polynomial and R is essentially smooth algebra over an infinite field k, or R is a regular ring over a perfect field k.

1. Introduction

One of the fundamental problems in commutative algebra, over last forty years, has been the following conjecture of M.P. Murthy ([10], [6, pp. 85]), on complete intersections in affine spaces, as follows.

Conjecture 1.1. *(See Murthy [10,6].) Suppose $A = k[X_1, X_2, \ldots, X_n]$ is a polynomial ring over a field k. Then, for any ideal I in A, $\mu(I) = \mu(I/I^2)$,* where μ denotes the minimal number of generators*.*

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The conjecture (1.1) is sometimes referred to as Murthy's complete intersection conjecture, because if I/I^2 is free then it means that I would be a complete intersection ideal.

Recently, Fasel [3] settled this conjecture (*with significant contributions from this author* [see e. g. [3, Lemma 3.1.2]]), affirmatively, when k is an infinite perfect field, with $1/2 \in k$. In this article, we do the same when k is an infinite field, with $1/2 \in k$. In fact, we prove a much stronger theorem (1.3), given below. A companion to Murthy's conjecture would be the following open problem.

Open Problem 1.2. Suppose $A = R[X]$ is a polynomial ring over a Noetherian commutative ring R. Suppose I is an ideal in A that contains a monic polynomial. Is $\mu(I) = \mu(I/I^2)$?

For such an ideal I, as in (1.2), when $\mu(I/I^2) \geq \dim(A/I) + 2$, Mohan Kumar ([9]) proved that I is image of a projective A-module of rank $\mu(I/I^2)$ and it was proved in [7] that $\mu(I) = \mu(I/I^2)$. For an ideal I, as in (1.2), without any further conditions, it was proved in [8], that I is set theoretically generated by $\mu(I/I^2)$-elements.

We settle this open problem (1.2), affirmatively, when R is a regular ring, as specified below (1.3).

Theorem 1.3. *Let R be a regular ring containing an infinite field k, with $1/2 \in k$. Assume R is essentially smooth over k or k is perfect. Suppose $A = R[X]$ is the polynomial ring and I is an ideal in A that contains a monic polynomial. Then, $\mu(I) = \mu(I/I^2)$. In fact, for $n \geq 2$, any set of n-generators of I/I^2 lifts to a set of generators of I.*

It was also well known that the validity of Murthy's conjecture would have implications on the renowned epi-morphism problem (see 1.4) of S. Abhayankar [1]. For an exposition of the same the readers are referred to [2], which would be our main reference on this. We state the epi-morphism problem from [2, Question 2.1], as follows.

Open Problem 1.4 *(S. Abhayankar).* Suppose k is a field, with $char(k) = 0$. Let

$$\varphi : k[X_1, X_2, \ldots, X_n] \longrightarrow k[Y_1, \ldots, Y_m] \quad \text{be an epi-morphism of } k\text{-algebras}$$

and $I = \ker(\varphi)$. Then, I is generated by $n - m$ variables. That means $I = (F_1, \ldots, F_m)$ for some $F_1, \ldots, F_{n-m} \in I$ and

$$k[X_1, X_2, \ldots, X_n] = k[F_1, \ldots, F_{n-m}, Y'_1, \ldots, Y'_m].$$

A weaker version of the epi-morphism problem would be the following conjecture (see [2, Question 2.2]).

Conjecture 1.5 *(S. Abhayankar). Suppose k is a field, with $char(k) = 0$ and*

$$\varphi : k[X_1, X_2, \ldots, X_n] \longrightarrow k[Y_1, \ldots, Y_m] \quad \textit{is an epi-morphism of } k\textit{-algebras}$$

and $I = \ker(\varphi)$. Then, $\mu(I) = n - m$.

As was indicated in [2], very limited progress has been made on the Problem 1.4. Note that Conjecture 1.5 is subsumed by Murthy's Conjecture 1.1 and same is true regarding the progress (see [2]). A much stronger theorem, than the Conjecture 1.5, follows from (1.3) for polynomial algebras over regular rings R, as specified below (1.6).

Theorem 1.6. *Let R be a regular ring over an infinite field k, with $1/2 \in k$. Assume R is essentially smooth over k or k is perfect. Suppose*

$$\varphi : R[X_1, X_2, \ldots X_n] \longrightarrow R[Y_1, Y_2, \ldots Y_m] \quad \textit{is an epi-morphism}$$

of polynomial R-algebras and $I = \ker(\varphi)$. If $n - m \geq \dim R + 1$, then $\mu(I) = \mu(I/I^2)$. In particular, if R is local, then I is a complete intersection ideal.

Note that the hypotheses in (1.3) entails only finite data. So, when k is perfect, using the theorem of Popescu [11], for the purposes of the proofs of (1.3), we would be able to assume that R is a smooth affine algebra over an infinite field (see the arguments in the proof of [13, Theorem 2.1]). With this approach, results in this article (e.g. 3.9) would be an improvement upon the respective versions in [3], relaxing the hypotheses in [3] that the rings are of essentially finite type over infinite perfect fields. Other than that, we also give an alternate description (5.2) of the obstruction set $\pi_0(Q_{2n})(A)$, defined in [3].

While the proof of Murthy's conjecture in [3] is elegant, its simplicity is even more astonishing. Proofs are further simplified, in this article, by proving that, for ideals I in a polynomial ring $A = R[X]$, that contains a monic polynomial, any local orientation is homotopically trivial (4.1). While reworking some of the proofs in [3], I also tried to elaborate the proof of [3, Theorem 1.0.5], and avoided any repetitions that would be unwarranted.

We comment on the organization of this article. In §2, we set up some notations and recall some of the definitions. To avoid the stronger hypotheses in [3], in §3, we restate and rework some of the results in [3]. In this section (§3), we also record a statement of the homotopy lifting property theorem (3.7), due to this author (unpublished), that was used in [3] and in (3.8). In §4 we prove the main theorem (1.3), in this article. In subsection §4.1, we summarize the main consequences of (1.3), including the solutions to Murthy's conjecture (1.1) and the weaker epi-morphism conjecture (1.5). In §5, we provide an alternate description of the obstruction set $\pi_0(Q_{2n})(A)$.

2. The obstruction presheaf

First, we establish some notations that will be useful throughout this article.

Notations 2.1. Throughout, k will denote a field (or ring), with $1/2 \in k$ and A, R will denote commutative Noetherian rings. For a commutative ring A and a finitely generated A-module M, the minimal number of generators of M will be denoted by $\mu(M)$.

We denote

$$q_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z^2, \qquad \tilde{q}_{2n+1} = \sum_{i=1}^{n} X_i Y_i + Z(Z-1).$$

Denote

$$Q_{2n} = \operatorname{Spec}(\mathscr{A}_{2n}) \text{ where } \mathscr{A}_{2n} = \frac{k[X_1, \dots, X_n, Y_1, \dots, Y_n, Z]}{(\tilde{q}_{2n+1})} \tag{1}$$

and

$$Q'_{2n} = \operatorname{Spec}(\mathscr{B}_{2n}) \text{ where } \mathscr{B}_{2n} = \frac{k[X_1, \dots, X_n, Y_1, \dots, Y_n, Z]}{(q_{2n+1} - 1)}. \tag{2}$$

There are inverse isomorphisms $\alpha : \mathscr{A}_{2n} \xrightarrow{\sim} \mathscr{B}_{2n}$ $\beta : \mathscr{B}_{2n} \xrightarrow{\sim} \mathscr{A}_{2n}$ given by

$$\begin{cases} \alpha(x_i) = \frac{x_i}{2} & 1 \le i \le n \\ \alpha(y_i) = \frac{y_i}{2} & 1 \le i \le n \\ \alpha(z) = \frac{z+1}{2} \end{cases} \qquad \begin{cases} \beta(x_i) = 2x_i & 1 \le i \le n \\ \beta(y_i) = 2y_i & 1 \le i \le n \\ \beta(z) = 2z - 1 \end{cases} \tag{3}$$

Therefore, $Q_{2n} \cong Q'_{2n}$. For a quadratic form q, of rank n, over a field k, $O(A, q) \subseteq GL_n(A)$ would denote the q-orthogonal subgroup and $EO(A, q) \subseteq O(A, q)$ would denote the elementary orthogonal subgroup. The category of schemes over $\operatorname{Spec}(k)$ will be denoted by $\underline{\mathrm{Sch}}_k$. Also, $\underline{\mathrm{Sets}}$ will denote the category of sets.

The homotopy presheaf $\pi_0(Q_{2n})$ of sets was proved to be a key tool in [3], which we recall next. Recall, a contravariant functor $\mathcal{F} : \underline{\mathrm{Sch}}_k \to \underline{\mathrm{Sets}}$ is also called a presheaf.

Definition 2.2. Given a presheaf $\mathcal{F} : \underline{\mathrm{Sch}}_k \to \underline{\mathrm{Sets}}$, and a scheme $X \in \underline{\mathrm{Sch}}_k$, define $\pi_0(\mathcal{F})(X)$ by the pushout

$$\begin{array}{ccc} \mathcal{F}(X \times \mathbb{A}^1) & \xrightarrow{T=0} & \mathcal{F}(X) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \mathcal{F}(X) & \longrightarrow & \pi_0(\mathcal{F})(X) \end{array} \quad \text{in } \underline{\mathrm{Sets}} \tag{4}$$

Note that $\pi_0(\mathcal{F})$ is also a presheaf of sets.

For an affine scheme $X = \operatorname{Spec}(A)$ and a presheaf $\mathcal{F}$, as above, we write $\mathcal{F}(A) := \mathcal{F}(\operatorname{Spec}(A))$ and $\pi_0(\mathcal{F})(A) := \pi_0(\mathcal{F})(\operatorname{Spec}(A))$. So, $\pi_0(\mathcal{F})(A)$ is given by the pushout diagram

$$\begin{array}{ccc} \mathcal{F}(A[T]) & \xrightarrow{T=0} & \mathcal{F}(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \mathcal{F}(A) & \longrightarrow & \pi_0(\mathcal{F})(A) \end{array} \quad \text{in } \underline{\text{Sets}} \tag{5}$$

Given a scheme $Y \in \underline{\text{Sch}}_k$, the association $X \mapsto \mathcal{H}om(X,Y)$ is a presheaf on $\underline{\text{Sch}}_k$. This presheaf is often identified with Y, itself. So, in some literature one may write, Y for the presheaf $\mathcal{H}om(-,Y)$ and $Y(X) := \mathcal{H}om(X,Y)$. Most importantly for us, for schemes $X, Y \in \underline{\text{Sch}}_k$, the pre-sheaves $\pi_0(Y)(X)$ are defined as in diagram (4), or (5). For the purposes of this article, $\pi_0(Q_{2n})(X)$ and $\pi_0(Q'_{2n})(X)$ would be of our particular interest. For $X = \text{Spec}(A)$, it follows immediately that, $Q_{2n}(A)$ and $Q'_{2n}(A)$ can be identified with the sets, as follows:

$$Q_{2n}(A) = \left\{(f_1,\ldots,f_n; g_1,\ldots,g_n; s) \in A^{2n+1} : \sum_{i=1}^{n} f_i g_i + s(s-1) = 0\right\}$$

$$Q'_{2n}(A) = \left\{(f_1,\ldots,f_n; g_1,\ldots,g_n; s) \in A^{2n+1} : \sum_{i=1}^{n} f_i g_i + s^2 - 1 = 0\right\}$$

The homotopy presheaves are given by the pushout diagrams in $\underline{\text{Sets}}$:

$$\begin{array}{ccc} Q_{2n}(A[T]) & \xrightarrow{T=0} & Q_{2n}(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ Q_{2n}(A) & \longrightarrow & \pi_0(Q_{2n})(A) \end{array} \quad \text{and} \quad \begin{array}{ccc} Q'_{2n}(A[T]) & \xrightarrow{T=0} & Q'_{2n}(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ Q'_{2n}(A) & \longrightarrow & \pi_0(Q'_{2n})(A) \end{array}$$

The isomorphism $Q_{2n} \cong Q'_{2n}$, induces a bijection $\pi_0(Q_{2n})(A) \cong \pi_0(Q'_{2n})(A)$.

For any ring A and $\mathbf{v} = (f_1,\ldots,f_n; g_1,\ldots,g_n; s) \in Q_{2n}(A)$, let us denote by $I(\mathbf{v})$ the ideal $(f_1,\ldots,f_n,s)A$. Also, let $\omega_{\mathbf{v}} : A^n \to \frac{I(\mathbf{v})}{I(\mathbf{v})^2}$ denote the surjective homomorphism defined by $e_i \mapsto f_i + I^2$ where $e_1,\ldots,e_n$ is the standard basis of A^n. Sometimes, $\omega_{\mathbf{v}}$ may be called a local orientation.

Now, we define local orientations of an ideal and the obstruction classes in $\pi_0(Q_{2n})$.

Definition 2.3. Suppose A is a commutative ring and I is an ideal in A. For an integer $n \geq 1$, a surjective homomorphism $\omega : A^n \twoheadrightarrow I/I^2$ would be called a local n-orientation of I. Clearly, a local n-orientation is determined by any set of elements $f_1,\ldots,f_n \in I$ such that $I = (f_1,\ldots,f_n) + I^2$. Further, given such a set of generators $f_1,\ldots,f_n$ of I/I^2, by Nakayama's lemma, there is an $s \in I$ such that $(1-s)I \subseteq (f_1,\ldots,f_n)A$, and hence $\sum_{i=1}^n f_i g_i + s(s-1) = 0$ for some $g_1,\ldots,g_n \in A$. Note,

$$(f_1,\ldots,f_n; g_1,\ldots,g_n; s) \in Q_{2n}(A).$$

Write

$$\zeta(I,\omega) := [(f_1,\ldots,f_n; g_1,\ldots,g_n,s)] \in \pi_0\left(Q_{2n}(A)\right)$$

It was established in [3, Theorem 2.0.7], that this association is well defined. We refer to $\zeta(I,\omega)$, as an obstruction class. Therefore, we have a commutative diagram

$$\begin{array}{ccc} Q_{2n}(A) & & \\ {\scriptstyle\eta}\downarrow & \searrow^{\zeta} & \\ \mathcal{O}(A,n) & \xrightarrow[\zeta]{} & \pi_0\left(Q_{2n}(A)\right) \end{array}$$

where $\mathcal{O}(A,n) =$ the set of all n-orientations (I,ω)

and $\eta(\mathbf{v}) = (I(\mathbf{v}), \omega_{\mathbf{v}})$. Note that we use the same notation ζ for two set theoretic maps.

3. Homotopy and the lifting property

In this section, we restate and rework, under the relaxed hypotheses in this article, some of the results in [3] on the homotopy lifting property of the local orientations, to point out the modifications needed. First, we quote and interpret the theorem [12, Theorem 1.3].

Theorem 3.1. *Suppose A is a regular ring containing a perfect field k. Let G be a reductive group scheme over k such that every semi-simple normal subgroup of G contains $\mathbb{G}_m^2$. Let $E(A)$ denote the corresponding elementary subgroup* (see [12, §2])*. Then,*

$$\forall\ \sigma(T) \in G(A[T]), \quad \sigma(0) = 1 \Longrightarrow \sigma(T) \in E(A[T]).$$

In particular, with $G = O(q_{2n+1})$, we have

$$\forall\ \sigma(T) \in O(A[T], q_{2n+1}) \subseteq GL_n(A[T]), \quad \sigma(0) = 1 \Longrightarrow \sigma(T) \in EO(A[T], q_{2n+1}).$$

Corollary 3.2. *Suppose A is a regular ring containing a field k. Then,*

$$\forall\ \sigma(T) \in O(A[T], q_{2n+1}), \quad \sigma(0) = 1 \Longrightarrow \sigma(T) \in EO(A[T], q_{2n+1}).$$

Proof. We will be following the arguments in [13, Theorem 2.1], to reduce the problem to the perfect field case. Let $\mathbb{F}$ be the prime subfield of k. By including the coefficients of the entries of $\sigma(T)$, there is a finitely generated subalgebra $A' \subseteq A$ over $\mathbb{F}$ such that $\sigma(T) \in A'[T]$. Note, $\sigma(T) \in O(A'[T], q_{2n+1})$ and $\sigma(0) = 1$. Note $\mathbb{F} \to A$ is geometrically regular (as defined in [13]). By Popescu's theorem [13, Corollary 1.2], we have the diagram

$$\begin{array}{ccccc} \mathbb{F} & \longrightarrow & A' & \longrightarrow & B \\ & \searrow & & \searrow & \downarrow \\ & & & & A \end{array} \quad \text{of homomorphisms}$$

such that B is smooth over $\mathbb{F}$. Since $\mathbb{F}$ is perfect, by Theorem 3.1, the $image(\sigma(T))$ in $B[T]$ is in $EO(B[T], q_{2n+1})$. Therefore, $\sigma(T) \in EO(A[T], q_{2n+1})$. The proof is complete. $\square$

Remark 3.3. The method of reduction to the perfect field case, in the proof of (3.2), using Popescu's theorem [13, Corollary 1.2] has the same flavor of the similar reduction in [14, pp. 507], due to Mohan Kumar, which is more elementary. One can work out an alternate proof of (3.2), using this argument of Mohan Kumar.

For the benefit of the readers, we would elaborate the proof of [3, Theorem 1.0.5], while reworking without the perfectness condition in [3]. Note that the orthogonal group $O(A, q_{2n+1})$ acts on $Q'_{2n}(A)$ in a natural way and therefore the elementary orthogonal group $EO(A, q_{2n+1})$ also acts on $Q'_{2n}(A)$. We will denote the set of orbits of the $EO(A, q_{2n+1})$-action by $\frac{Q'_{2n}(A)}{EO(A,q_{2n+1})}$.

Theorem 3.4. *Let A be a essentially smooth algebra over an infinite field k, with $1/2 \in k$. Then, for $n \geq 2$, the natural map*

$$\varphi : \frac{Q'_{2n}(A)}{EO(A, q_{2n+1})} \longrightarrow \pi_0\left(Q'_{2n}\right)(A) \quad \textit{is a bijection.}$$

Proof. Clearly, φ is well defined and is surjective. Now suppose $H(T) \in Q'_{2n}(A[T])$ is a homotopy. We need to show that there is a matrix $\tau \in EO(A, q_{2n+1})$ such that $H(0) = H(1)\tau$. For $R = \mathscr{B}_{2n}, A[T], A$, use the following generic notations, to denote the quadratic modules

$$\begin{cases} q := q_{2n+1} : R^{2n+1} \to R & \text{sending} \quad (u_1, \ldots, u_n, v_1, \ldots, v_n, s) \mapsto \sum_{i=1}^n u_i v_i + s^2 \\ q_0 : R \to R & \text{sending} \quad s \mapsto s^2 \end{cases}$$

As usual, define $B_q(e, e') = \frac{q(e+e')-q(e)-q(e')}{2}$. With respect to the standard basis, the matrix of B_q is given by

$$B_q := \frac{1}{2}\begin{pmatrix} 0 & I_n & 0 \\ I_n & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

So, the map $R^{2n+1} \to (R^{2n+1})^*$ sends $\mathbf{v} \mapsto \mathbf{v}B_q$. These bilinear forms give the following exact sequences (write $\mathscr{B} := \mathscr{B}_{2n}$)

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathscr{B}^{2n+1} \xrightarrow{\langle(\mathbf{x},\mathbf{y},z),-\rangle} \mathscr{B} \longrightarrow 0$$

$$0 \longrightarrow K \longrightarrow A[T]^{2n+1} \xrightarrow{\langle H(T),-\rangle} A[T] \longrightarrow 0$$

$$0 \longrightarrow K_0 \longrightarrow A^{2n+1} \xrightarrow[\langle H(0),-\rangle]{} A \longrightarrow 0$$

$$\text{So, } \mathcal{K} = (\mathscr{B}(\mathbf{x},\mathbf{y},z))^{\perp}, \quad K = (A[T]H(T))^{\perp}, \quad K_0 = (AH(0))^{\perp}$$

are orthogonal complements, which inherit the quadratic structures. It follows $\overline{K} := K \otimes \frac{A[T]}{(T)} = (RH(0))^{\perp} \cong K_0$. Therefore, there is an isometry, $\sigma_0 : K_0 \xrightarrow{\sim} \overline{K}$, which extends to $\sigma_0 \otimes A[T] : K_0 \otimes A[T] \xrightarrow{\sim} \overline{K} \otimes A[T]$, an isometry. (*For clarity, note that it follows from Lindel's theorem ([5]) that there is an isomorphism* $\overline{K} \otimes A[T] \xrightarrow{\sim} K$, *which need not be an isometry.*) We claim that K is extended from A. The question is local (see [4, 5.3.2]). In the local case, K is extended from A, by [15, Theorem 3.1]. This is because K corresponds to a $O(q_{2n}, A[T])$-torsor, where $q_{2n} = \sum_{i=1}^{n} X_iY_i$ is the hyperbolic quadratic form. In deed, K corresponds to a Zariski locally trivial torsor. This establishes the claim (see [3] for further details).

Hence, there is an isometry $\overline{K} \otimes A[T] \xrightarrow{\sim} K$. By this identification, we say σ_0 extends to an isometry $\sigma_0 \otimes A[T] : K_0 \otimes A[T] \xrightarrow{\sim} K$. Also, note

$$(A[T]H(T), q_{|A[T]H(T)}) \cong (A[T], q_0) \cong (A[T]H(0), q_{|A[T]H(0)})$$

Putting all these together, there is an isometry $\sigma(T) \in O\,(A[T], q)$ such that the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K_0 \otimes A[T] & \longrightarrow & A[T]^{2n+1} & \xrightarrow{\langle H(0),-\rangle} & A[T] & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \sigma_0\otimes 1} & & \downarrow{\scriptstyle \sigma(T)} & & \| & & \\ 0 & \longrightarrow & K & \longrightarrow & A[T]^{2n+1} & \xrightarrow[\langle H(T),-\rangle]{} & A[T] & \longrightarrow & 0 \end{array} \tag{6}$$

commutes. By construction, (*alternately, by composing with* $\sigma(0)^{-1} \otimes Id$), we have $\sigma(0) = I_{2n+1}$. Now, by Corollary 3.2, it follows $\sigma(T) \in EO(A[T])$ and hence $\sigma(1) \in EO(A)$. Since $H(T)\sigma(T) = H(0)$, the proof is complete. □

Before we proceed, we define the action of $EO\,(A, q_{2n+1})$ on $Q_{2n}(A)$ and give another definition, for the convenience of subsequent discussions.

Definition 3.5. Fix a commutative ring A. As usual, $EO\,(A, q_{2n+1})$ acts on A^{2n+1}, which restricts to an action on $Q'_{2n}(A)$. Using the correspondences $\alpha : Q_{2n}(A) \xrightarrow{\sim} Q'_{2n}(A)$, $\beta : Q'_{2n}(A) \xrightarrow{\sim} Q_{2n}(A)$, define an action on $Q_{2n}(A)$ as follows:

$$\forall\ \mathbf{v} \in Q_{2n}(A), M \in EO\,(A, q_{2n+1}) \quad \text{define} \quad \mathbf{v} * M := \beta\,(\alpha(\mathbf{v})M)$$

This action is not given by the usual matrix multiplication. Five different classes of the generators of $EO\,(A, q_{2n+1})$ and their actions on $Q_{2n}(A)$ are given in [3].

Definition 3.6. Let A be a commutative ring over k. Let $\mathbf{v} \in Q_{2n}(A)$. We write $\mathbf{v} := (a_1, \ldots, a_n; b_1, \ldots, b_n; s)$. For integers, $r \geq 1$ we say that r-lifting property holds for $\mathbf{v}$, if

$$I(\mathbf{v}) = (a_1 + \mu_1 s^r, \ldots, a_n + \mu_n s^r) \qquad \text{for some } \mu_i \in A.$$

We say the lifting property holds for $\mathbf{v}$, if

$$I(\mathbf{v}) = (a_1 + \mu_1, \ldots, a_n + \mu_n) \qquad \text{for some } \mu_i \in I(\mathbf{v})^2.$$

Before we allude to the key result in [3, Corollary 3.2.6] (see (3.8)), we record the following homotopy lifting theorem, due to this author (unpublished), that was used crucially in the proof.

Theorem 3.7. *Let R be a regular ring containing a field k. Let $H(T) := (f_1(T), \ldots, f_n(T), g_1(T), \ldots, g_n(T), s) \in Q_{2n}(R[T])$, with $s \in R$. Write $a_i = f_i(0), b_i = g_i(0)$. Write $I(T) = (f_1(T), \ldots, f_n(T), s)$. Also assume $I(0) = (a_1, \ldots, a_n)$. Then,*

$$I(T) = (F_1, \ldots, F_n) \quad \ni \quad f_i - F_i \in s^2 R[T]$$

Proof. See [3, Lemma 3.1.2]. □

Theorem 3.8. *Suppose A is a regular ring containing a field k, with $1/2 \in k$. Let $n \geq 2$ be an integer. Let $\mathbf{v} \in Q_{2n}(A)$ and $M \in EO\,(A, q_{2n+1})$. Then, $\mathbf{v}$ has 2-lifting property if and only if $\mathbf{v} * M$ has the 2-lifting property.*

Proof. We outline the proof in [3]. It would be enough to assume that M is a generator of $EO\,(A, q_{2n+1})$. There would be five cases to deal with, one for each type of generators of $EO\,(A, q_{2n+1})$, listed in [3, pp. 3–4]. One of them, that is of the case of generators of the type 4 (in the list [3, pp. 3–4]), is fairly involved. This case follows from Theorem 3.7 (see [3, Lemma 3.1.2]). □

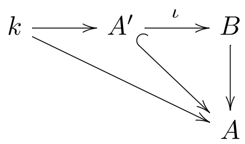
The following summarizes the final results on homotopy and lifting of generators (also see [3, Theorem 3.2.7]).

Theorem 3.9. *Suppose A is a regular ring containing an infinite field k, with $1/2 \in k$. Assume A is essentially smooth over k or k is perfect. Let $n \geq 2$ be an integer. Denote $\mathbf{0} := (0, \ldots, 0; 0, \ldots, 0; 0) \in Q_{2n}(A)$ and let $\mathbf{v} \in Q_{2n}(A)$. Then, the following conditions are equivalent:*

1. *The obstruction* $\zeta\left(I(\mathbf{v},\omega_{\mathbf{v}})\right) = [\mathbf{0}] \in \pi_0\left(Q_{2n}\right)(A)$.
2. $\mathbf{v}$ *has* 2*-lifting property.*
3. $\mathbf{v}$ *has the lifting property.*
4. $\mathbf{v}$ *has* r*-lifting property,* $\forall\, r \geq 2$.

Proof. It is clear, (2) $\Longrightarrow$ (3). To prove (3) $\Longrightarrow$ (1), suppose $I(\mathbf{v}) = (a_1+\mu_1, \ldots, a_n+\mu_n)$, with $\mu_i \in I(\mathbf{v})^2$. Write $\mathbf{v}' = (a_1+\mu_1, \ldots, a_n+\mu_n; 0, \ldots, 0; 0) \in Q_{2n}(A)$. By [3, 2.0.10], we have $\zeta\left(I(\mathbf{v},\omega_{\mathbf{v}})\right) = \zeta\left(I(\mathbf{v}',\omega_{\mathbf{v}'})\right) = [v_0] \in \pi_0\left(Q_{2n}\right)$. This establishes, (3) $\Longrightarrow$ (1).

Now we prove (1) $\Longrightarrow$ (2). Assume $\zeta\left(I(\mathbf{v},\omega_{\mathbf{v}})\right) = [\mathbf{0}]$. In case A is essentially finite over k, it follows from Theorem 3.4 that $\mathbf{0} = \mathbf{v} * M$, for some $M \in EO(A, q_{2n+1})$ and (2) follows from Theorem 3.8. However, when A is regular and contains an infinite perfect field, we have to use Popescu's theorem. By definition, $\zeta\left(I(\mathbf{v},\omega_{\mathbf{v}})\right) = [\mathbf{0}]$ implies that there is a chain homotopy from $\mathbf{v}$ to $\mathbf{0}$. This data can also be encapsulated in a finitely generated algebra A' over k. As in the proof of (3.2) there is a diagram



of homomorphisms

such that B is smooth over k. The homotopy relations are carried over to B. Therefore, by replacing A by B, we can assume that A is essentially smooth over k. So, Theorem 3.4 applies and (2) follows as in the previous case.

So, it is established that (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3). It is clear that (4) $\Longrightarrow$ (2). Now suppose, one of the first three conditions hold. Fix $r \geq 2$. Notice $I(\mathbf{v}) = (a_1, \ldots, a_n, s^r)A$. So, replacement of s by s^r leads to the same obstruction class in $\in \pi_0\left(Q_{2n}\right)(A)$, which is $= [\mathbf{0}] \in \pi_0\left(Q_{2n}\right)(A)$. Since (1) $\Longleftrightarrow$ (2), it follows $I(\mathbf{v})$ has $2r$-lifting property and hence the r-lifting property. The proof is complete. □

4. Monic polynomials and the lifting property

In this section, first we give an application of (3.9), for ideals containing monic polynomials. In fact, we prove that all local orientations of such ideals have trivial obstruction class, as follows.

Proposition 4.1. *Suppose* $R = A[X]$ *is a polynomial ring over a commutative ring* A *and* I *is an ideal that contains a monic polynomial. Suppose* $\omega : R^n \twoheadrightarrow I/I^2$ *is a surjective homomorphism (local orientation). Then* $\zeta(I,\omega) = [\mathbf{0}] \in \pi_0\left(Q_{2n}\right)(R)$*, where* $\mathbf{0} := (0, 0, \ldots, 0, 0, \ldots, 0) \in Q_{2n}(R)$.

Proof. Let $f_1, \ldots, f_n \in I$ be a lift of ω. Then, $I = (f_1, f_2, \ldots, f_n) + I^2$. We can assume that f_1 is a monic polynomial, with even degree. Now, consider the transformation [7]:

$$\varphi : A[X, T^{\pm 1}] \xrightarrow{\sim} A[X, T^{\pm 1}] \quad \text{by} \quad \begin{cases} \varphi(X) = X - T + T^{-1} \\ \varphi(T) = T \end{cases}$$

There is a commutative diagram

$$\begin{array}{ccc} A[X] & = & A[X] \\ \downarrow & & \uparrow{\scriptstyle T=1} \\ A[X, T^{\pm 1}] & \xrightarrow[\varphi]{} & A[X, T^{\pm 1}] \end{array}$$

Then, $\varphi(f_1) = f_1(X - T + T^{-1})$ is doubly monic in T, meaning that its lowest and the highest degree terms have coefficients 1. Let $F_1(X, T) = T^{\deg f_1(X)} \varphi(f_1) \in A[X, T]$. Then, $F_1(X, 0) = 1$. Also, for $i = 2, \ldots, n$ write $F_i(X, T) = T^{\delta} \varphi(f_i)$, for some integer $\delta \gg 0$, such that $F_i(X, T) \in TA[X, T]$. Therefore, $F_i(X, 0) = 0$. Now, write

$$\mathscr{I}' = \varphi(IA[X, T^{\pm 1}]) \quad \text{and} \quad \mathscr{I} := \mathscr{I}' \cap A[X, T].$$

Since $\frac{A[X,T]}{\mathscr{I}} \xrightarrow{\sim} \frac{A[X,T^{\pm 1}]}{\mathscr{I}'}$, it follows

$$\mathscr{I} = (F_1(X, T), \ldots, F_n(X, T)) + \mathscr{I}^2.$$

Therefore, by Nakayama's Lemma, there is a $S(X, T) \in \mathscr{I}$, such that

$$(1 - S(X, T))\mathscr{I} \subseteq (F_1(X, T), F_2(X, T), \ldots, F_n(X, T))$$

and hence

$$\sum F_i(X, T)G_i(X, T) + S(X, T)(S(X, T) - 1) = 0$$

for some $G_1, \ldots, G_n \in A[X, T]$. Write

$$\psi(X, T) = (F_1(X, T), F_2(X, T), \ldots, F_n(X, T); G_1(X, T), \ldots, G_n(X, T); S(X, T))$$

Then, $\psi(X, T) \in Q_{2n}(A[X, T])$ and $\mathscr{I}_{|T=1} = I$. Further,

$$\psi(X, 1) = (f_1, \ldots, f_n; G_1(X, 1), \ldots, G_n(X, 1); S(X, 1))$$

and

$$\psi(X, 0) = (1, 0, \ldots, 0; G_1(X, 0), \ldots, G_n(X, 0), S(X, 0)).$$

By [3, 2.0.10], $\psi(X, 0) \sim \mathbf{0} \in Q_{2n}(R)$. Hence, $\psi(X, 1) \sim \mathbf{0} \in Q_{2n}(R)$. Therefore,

$$\zeta(I, \omega) = [\psi(X, 1)] = [\mathbf{0}] \in \pi_0\left(Q_{2n}(R)\right).$$

The proof is complete. □

The following is the main theorem in this article, which is an extension of the main theorem in [7] mentioned in the introduction.

Theorem 4.2. *Let R be a regular ring over an infinite field k, with $1/2 \in k$ and $A = R[X]$ is the polynomial ring. Assume R is essentially smooth over k or k is perfect. Suppose, I is an ideal in A that contains a monic polynomial. Then $\mu(I) = \mu(I/I^2)$. In fact, if $\mu(I/I^2) \geq 2$, any local orientation $\omega : A^n \twoheadrightarrow I/I^2$ lifts to a set of generators of I.*

Proof. If $\mu(I/I^2) = 1$, then I is an invertible ideal with a monic polynomial, hence it is free. The rest follows immediately from Proposition 4.1 and Theorem 3.9. The proof is complete. □

4.1. The consequences

In this subsection, we summarize some of the consequences of the monic polynomial Theorem 4.2. First, the following is the statement on the solution of the conjecture of M.P. Murthy (1.1). The theorem is due to Fasel [3, Theorem 3.2.9], in the case when k is perfect. However, our proof is much direct.

Theorem 4.3. *Let k be an infinite field, with $1/2 \in k$. Let $A = k[X_1, \ldots, X_n]$ be the polynomial ring. Then, for any ideal I of A, $\mu(I) = \mu(I/I^2)$.*

Proof. By a change of variables (see [6, Theorem 6.1.5]), we can assume that I contains a monic polynomial in X_n. Now, the proof is complete by (4.2). □

Following would be a more general version of Theorem 4.3.

Theorem 4.4. *Let R be a regular ring over an infinite field k, with $1/2 \in k$ and $A = R[X_1, X_2, \ldots, X_n]$ be the polynomial ring in n variables. Assume R is essentially smooth over k or k is perfect. Suppose, I is an ideal in A with height$(I) \geq \dim R + 1$. Then $\mu(I) = \mu(I/I^2)$. In fact, if $\mu(I/I^2) \geq 2$, any local orientation $\omega : A^n \twoheadrightarrow I/I^2$ lifts to a set of generators of I.*

Proof. Again, by a change of variables (see [6, Theorem 6.1.5]), we can assume that I contains a monic polynomial in X_n. Now, the proof is complete by (4.2). □

The following encompasses the solution of the weaker version of S. Abhyankar's epimorphism conjecture (1.5), in the case when k is an infinite field.

Theorem 4.5. *Let R be a regular ring over an infinite field k, with $1/2 \in k$. Assume R is essentially smooth over k or k is perfect. Suppose*

$$\varphi : R[X_1, X_2, \dots X_n] \longrightarrow R[Y_1, Y_2, \dots Y_m] \quad \textit{is an epi-morphism}$$

of polynomial R-algebras and $I = \ker(\varphi)$. *If* $n - m \geq \dim R + 1$, *then* $\mu(I) = \mu(I/I^2)$. *In particular, if R is local, then I is a complete intersection ideal.*

Proof. Since, $height(I) = n - m \geq \dim R + 1$, it follows from (4.4) that $\mu(I) = \mu(I/I^2)$. Note I/I^2 is a projective $R[Y_1, Y_2, \dots Y_m]$-module of rank $n - m$. If R is local, then I/I^2 is free of rank $n - m$ (see [13, Theorem 2.1]). Hence, $\mu(I) = \mu(I/I^2) = n - m$. So, I is a complete intersection ideal. The proof is complete. □

The following is the statement on the solution of the weaker version of S. Abhyankar's epimorphism conjecture (1.5).

Corollary 4.6. *Suppose k is an infinite field, with* $1/2 \in k$ *and*

$$\varphi : k[X_1, X_2, \dots X_n] \longrightarrow k[Y_1, Y_2, \dots Y_m] \quad \textit{is an epi-morphism}$$

of polynomial rings. Then, I is generated by $n - m$ *elements.*

Proof. Follows immediately from (4.5). □

5. Alternate obstructions

In this section, we give an alternate description of the obstruction sheaf $\pi_0(Q_{2n})(A)$, which appears more traditional.

Definition 5.1. Suppose A is a commutative ring and $n \geq 1$ is an integer. Write

$$\mathcal{Q}_n(A) := \left\{ (f_1, \dots, f_n, s) \in A^{n+1} : \exists\, g_1, \dots, g_n \in A \ni \sum_{i=1}^{n} f_i g_i + s(s-1) = 0 \right\}$$

Note $A \mapsto \mathcal{Q}_n(A)$ is a presheaf on the category of affine schemes. As in diagram (4), one can define $\pi_0(\mathcal{Q}_n(A))$, by the pushout:

$$\begin{array}{ccc} \mathcal{Q}_n(A[T]) & \xrightarrow{T=0} & \mathcal{Q}_n(A) \\ {\scriptstyle T=1}\downarrow & & \downarrow \\ \mathcal{Q}_n(A) & \longrightarrow & \pi_0(\mathcal{Q}_n)(A) \end{array} \qquad \text{in } \underline{\text{Sets}}.$$

There is a natural map of sheaves $Q_{2n}(A) \to \mathcal{Q}_n(A)$ sending

$$(f_1, \dots, f_n; g_1, \dots, g_n; s) \mapsto (f_1, \dots, f_n, s)$$

This induces a surjective map $\Phi : \pi_0(Q_{2n})(A) \twoheadrightarrow \pi_0(\mathcal{Q}_n)(A)$.

Lemma 5.2. *The map* $\Phi : \pi_0(Q_{2n})(A) \twoheadrightarrow \pi_0(\mathcal{Q}_n)(A)$ *is a bijection.*

Proof. We define the inverse map $\Psi : \pi_0(\mathcal{Q}_P) \longrightarrow \pi_0(Q_{2n}(A))$. Let $(f_1, \ldots, f_n, s) \in \mathcal{Q}_n(A)$. Then, $\sum f_i g_i + s(s-1) = 0$ for some $g_i \in A$. We define,

$$\Psi([(f_1, \ldots, f_n, s)]) = [(f_1, \ldots, f_n; g_1, \ldots, g_n; s)] \in \pi_0(Q_{2n})(A).$$

We need to show that Ψ is well defined. With $I = (f_1, \ldots, f_n, s)$, let $\omega : A^n \twoheadrightarrow \frac{I}{I^2}$ be induced by $f_1, \ldots, f_n$. By [3, Lemma 2.0.10], the obstruction $\zeta(I, \omega_I) := [(f_1, \ldots, f_n; g_1, \ldots, g_n; s)] \in \pi_0(Q_{2n}(A))$ is independent of $g_1, \ldots, g_n$. Therefore, Ψ is well defined. It is now clear that

$$\Phi\Psi = Id \qquad \text{and} \qquad \Psi\Phi = Id.$$

The proof is complete. □

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