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On efficient generation of ideals

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0. Introduction

In this paper we shall discuss the question that *if R is a commutative noetherian ring and I is an ideal of R, then whether I is generated by $\mu(I/I^2)$ elements* (μ denotes minimal number of generators). In general it is known that $\mu(I/I^2) \leqq \mu(I) \leqq \mu(I/I^2)+1$ ([5], Lemma).

The Main result (Theorem 1.2) here is that *if $R=A[X]$ is a polynomial ring over a noetherian commutative ring and I is an ideal of R which contains a monic polynomial and if $\mu(I/I^2) \geqq \dim(R/I)+2$ then I is actually generated by $\mu(I/I^2)$ elements.* This result is an improvement of a result of Mohan Kumar ([5], page 161, Satz 5.18), which says that under the same conditions I is a quotient of a projective module of rank $\mu(I/I^2)$.

The final result in this paper is Theorem 2.2. The theorem says that *if $R = A[X_1, \ldots, X_n, T_1^{\pm 1}, \ldots, T_r^{\pm 1}]$, n and $r \geqq 0$ is a Laurent polynomial ring, with n polynomial variables and r Laurent polynomial variables, over a noetherian commutative ring A and if I is an ideal of R with $\mu(I/I^2) \geqq \dim(R/I)+2$ and* height $(I) > \dim A$, *then I is generated by $\mu(I/I^2)$ elements.*

When R is a polynomial ring over a field then this theorem is due to Mohan Kumar ([6], 4, Theorem 5). In the polynomial case i.e. when $r=0$ Theorem 2.2 is stated as Corollary 1.5.

In [4] we have proved Theorem 2.2 for $r>0$ ([4], Chapter III, Theorem 2.3). For $r=0$, the content of Corollary 1.5 was raised as a question ([4], Chapter III, Remark 2.6). S.M. Bhatwadekar and R.A. Rao ([2]) proved Corollary 1.5 in the case when A is affine domain.

Before we conclude this section we shall recall some standard notations.

Throughout this paper R and A will denote commutative noetherian rings with finite krull dimension. By $\dim A$ we shall mean the Krull dimension of A. For an R-module M, $\mu(M)$ will denote the minimal number of generators of M as an R-module.

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$R=A[X_1,\dots,X_n, T_1^{\pm 1},\dots, T_r^{\pm 1}]$, n and $r\geq 0$ will be the Laurent polynomial ring over A, with n polynomial variables $X_1,\dots,X_n$ and r Laurent polynomial variables $T_1,\dots,T_r$.

1. The main result

In this section we shall prove our main result (Theorem 1.2). Before we go in to our main discussion we introduce the following definitions.

Definition 1.1. A monic polynomial f in $A[X]$ is said to be a *special monic polynomial* if the constant term of f is equal to one. A Laurent polynomial f in $A[X,X^{-1}]$ is said to be *doubly monic Laurent polynomial* if both the coefficients of the highest degree term and the lowest degree term in f are equal to one.

So, a special monic polynomial is a doubly monic Laurent polynomial.

Theorem 1.2. *Let $R=A[X]$ be a polynomial ring over a commutative noetherian ring A and I an ideal of R. Suppose I contains a monic polynomial and $\mu(I/I^2)\geq \dim(R/I)+2$. Then $\mu(I)=\mu(I/I^2)$.*

In the proof of Theorem 1.2 we shall actually extend the ideal I to the Laurent polynomial extension $R[T, T^{-1}]=A[X,T,T^{-1}]$ and prove that $IR[T,T^{-1}]$ is generated by $\mu(I/I^2)$ elements. Following remark introduces a change of variable in $A[X,T,T^{-1}]$, which will be used in the proof of the theorem.

Remark 1.3. Let $R=A[X]$ be as in Theorem 1.2 and $R[T,T^{-1}]=A[X,T,T^{-1}]$ be its Laurent polynomial extension. Define an A-automorphism θ of $R[T,T^{-1}]$ as follows,

$$\theta(X)=X+T+T^{-1}$$
$$\theta(T)=T.$$

We observe that if f is a monic polynomial in R then $\theta(f)$ is doubly monic Laurent polynomial in T.

We state another lemma before we give the proof of the Theorem 1.2. This lemma is a slight variation of Mohan Kumar's lemma ([6], 3, Lemma 3) on prime avoidance and the proof is also similar.

Lemma 1.4. *Let A be a commutative noetherian ring and I, J be ideals of A, I containing J. Let $n=\mu(I/I^2)$. Assume that $a_1,\dots,a_r$; $r<n$ are elements of I. Further suppose,*

(i) *$a_1,a_2,\dots,a_r$ form a part of a minimal set of generators of $I \bmod I^2$.*

(ii) *Whenever P is a prime ideal of A which contains $(a_1A+\dots+a_rA)+J$ and does not contain I, the image of P in $A/(a_1A+J)$ has height at least d, for some fixed integer d.*

Then we can find a_{r+1} in I such that,

(i) *$a_1,\dots,a_{r+1}$ form a part of minimal set of generators of $I \bmod I^2$.*

(ii) *Whenever P is a prime ideal of A, which contains $(a_1A+\ldots+a_{r+1}A)+JA$ and does not contain I, the image of P in $A/(a_1A+J)$ has height atleast $d+1$.*

Proof of Theorem 1.2. Suppose a_1 belongs to a minimal set of generators of $I \bmod I^2$. Since I contains a monic polynomial f, replacing a_1 by a_1+f^p for large enough p, we can assume a_1 is monic.

Write $J=A\cap I$. Then $A/J \to R/I$ and $A/J \to R/(J, a_1)R$ are integral extensions. So we have $\dim(A[X]/I)=\dim(A/J)=\dim(R/(J, a_1)R)$.

Write $B=R/(J, a_1)R$. By Lemma 1.4 we can choose a_2 in I such that,

(i) a_1, a_2 form a part of minimal set of generators of $I \bmod I^2$.

(ii) If a prime ideal P of R contains a_1R+a_2R+JR and does not contain I, then image of P in B has height atleast one.

If we write $n=\mu(I/I^2)$, then by iterating the above process we can find $a_1, \ldots, a_n$ in I with a_1 monic and such that,

(i) $a_1, \ldots, a_n$ form a minimal set of generators of $I \bmod I^2$.

(ii) Whenever P is a prime ideal of R which contains $(a_1R+\ldots+a_nR)+JR$ and does not contain I, the image of P in B has height atleast $n-1$.

Since $n\geqq\dim(R/I)+2=\dim B+2$, by (ii) we have,

(iii) For a prime ideal P of R, if P contains $(a_1R+\ldots+a_nR)+JR$ then P also contains I.

Let $R'=A[X, T, T^{-1}]=R[T, T^{-1}]$ be the Laurent polynomial extension of R and θ be the A-automorphism defined in Remark 1.3. Since substitution $T=1$ gives a retraction of R' to R, it is enough to prove IR' is generated by n elements. Instead, we shall prove that $\theta(IR')$ is generated by n elements.

We shall write $I_1=\theta(IR')$, $I'=I_1\cap R[T]$ and $J'=I_1\cap R=I'\cap R$.

As $\theta(J)=J$ is contained in J', it follows from (i) and (iii) that,

(iv) $\theta(a_1), \ldots, \theta(a_n)$ generate $I_1 \bmod I_1^2$.

(v) For a prime ideal P in $\operatorname{Spec}(R')$, if P contains

$$(\theta(a_1)R'+\ldots+\theta(a_n)R')+J'R'$$

then P also contains I_1.

For $i=1$ to n if we write $f_i=T^{r_i}\theta(a_i)$, for some suitable r_i, then we can assume that f_1 is special monic in $R[T]$ (because $\theta(a_1)$ is doubly monic) and $f_2, \ldots, f_n$ belongs to $TA[X, T]=TR[T]$. Since T is a unit in R', we can replace $f_1, \ldots, f_n$ in condition (iv) and (v).

As prime ideals in $\operatorname{Spec}(R[T])$ which contains T can not contain f_1, it follows immidiately from (iv) and (v) that

(i′) $f_1, \ldots, f_n$ generate $I' \bmod I'^2$.

(ii′) For a prime ideal P in $\operatorname{Spec}(R[T])$, if P contains

$$(f_1R[T]+\ldots+f_nR[T])+J'R[T]$$

then P also contains I'.

(iii′) f_1 is special monic in T and $f_2, \ldots, f_n$ belongs to $TR[T]$.

We are going to prove that I' is generated by n elements in $R[T]$

Consider the multiplicative set $1+J'$ in R. Since J' is in the radical of $R_{1+J'}$ and $R_{1+J'}[T]/f_1$ is integral extension of $R_{1+J'}$, we have J' is also contained in the radical of $R_{1+J'}[T]/f_1$. In view of (ii′) a maximal ideal of $R_{1+J'}[T]/f_1$ which contains images of $f_2,\dots,f_n$, will also contain I'', the image of I' in $R_{1+J'}[T]/f_1$. And thus by (i′) for a maximal ideal M of $R_{1+J'}[T]/f_1$, which contains the image of $f_2,\dots,f_n$, we have I''_M is generated by the images of $f_2,\dots,f_n$ and hence I'' is generated by these elements. So it follows

$$I'_{1+J'}=f_1R_{1+J'}[T]+\dots+f_nR_{1+J'}[T].$$

Thus

$$I'_{1+s}=f_1R_{1+s}[T]+\dots+f_nR_{1+s}[T],$$

for some s in J'.

We shall assume that s is not nilpotent (otherwise I' is generated by n elements).

As a consequence the following sequence

$$0\to K\to R_{1+s}[T]^n\xrightarrow{(f_i)}I'_{1+s}\to 0$$

is exact, where K is the kernel of the obvious surjection defined by $f_1,\dots,f_n$. As s belongs to I', K_s is projective and since f_1 is monic polynomial, by Quillen-Suslin Theorem ([8], Theorem 3/[9], Theorem 1) K_s is free of rank $n-1$.

Since $I'_s=R_s[T]$ we have an exact sequence over $R_s[T]$

$$0\to K'\to R_s[T]^n\xrightarrow{(1,0,\dots,0)}I'_s\to 0$$

where the surjection is the obvious map defined by $1,0,\dots,0$ and K' is the kernel of the surjection which is free.

Let us denote "mod T" by "bar". Now as $f_1(0)=1$, $f_2(0)=0$, $f_n(0)=0$, there is an isomorphism $g\colon \bar{K}_s\to\bar{K}'_{1+s}$ such that the following diagram

$$\begin{array}{ccccccc}
0\to & \bar{K}_s & \longrightarrow & R^n_{s(1+s)} & \xrightarrow{(\bar{f}_i)} & \bar{I}'_{s(1+s)} & \to 0\\
 & \downarrow g & & \| & & \| & \\
0\to & \bar{K}'_{1+s} & \longrightarrow & R^n_{s(1+s)} & \xrightarrow{(1,0,\dots,0)} & \bar{I}'_{s(1+s)} & \to 0
\end{array}$$

is commutative, where the last and the middle vertical maps are identity.

Since K_s and K'_{1+s} are extended (infact free), there are isomorphisms

$$\beta\colon K_s\to\bar{K}_s\otimes R_{s(1+s)}[T]\quad\text{and}\quad\lambda\colon K'_{1+s}\to\bar{K}'_{1+s}\otimes R_{s(1+s)}[T]$$

which are identity modulo T. If we write $h=\lambda^{-1}\circ(g\otimes\mathrm{Id})\circ\beta$, then $h\colon K_s\to K'_{1+s}$ is an isomorphism. Also $\bar{h}=\bar{\lambda}^{-1}\circ g\circ\bar{\beta}=g$. Hence we have an isomorphism $h\colon K_s\to K'_{1+s}$ such that $\bar{h}=g$.

Using splittings of the surjections

$$R_{s(1+s)}[T]^n \to I'_{s(1+s)} \to 0 \quad \text{and} \quad R_{s(1+s)}[T]^n \to I'_{s(1+s)} \to 0$$

which are equal "modulo T", we can define an isomorphism

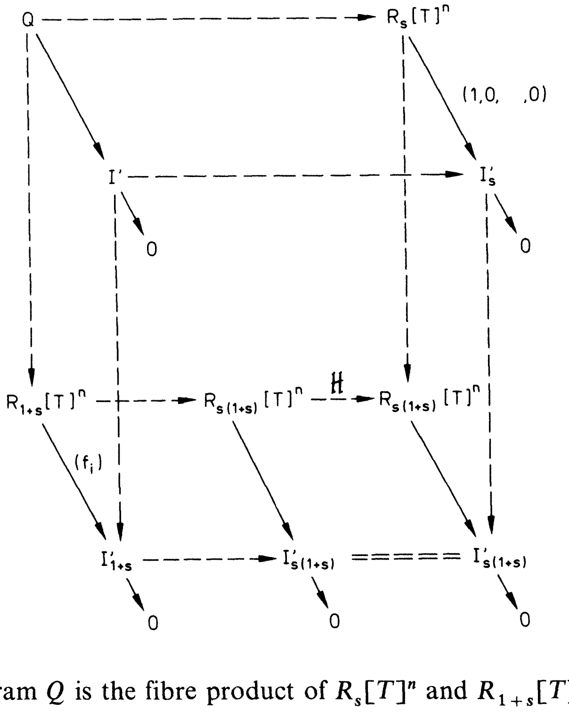
$$H\colon R_{s(1+s)}[T]^n \to R_{s(1+s)}[T]^n$$

such that $H \equiv \mathrm{Id} \pmod T$ and the following diagram

$$\begin{array}{ccccccc} 0 \to K_s & \longrightarrow & R_{s(1+s)}[T]^n & \xrightarrow{(f_i)} & I'_{s(1+s)} & \to 0 \\ \downarrow h & & \downarrow H & & \| \ \mathrm{Id} & \\ 0 \to K'_{1+s} & \longrightarrow & R_{s(1+s)}[T]^n & \xrightarrow{(1,0,\dots,0)} & I'_{s(1+s)} & \to 0 \end{array}$$

is commutative.

As $Rs + R(1+s) = R$, we can construct the following fibre product diagram,



In this diagram Q is the fibre product of $R_s[T]^n$ and $R_{1+s}[T]^n$ given by the maps

$$R_s[T]^n \to R_{s(1+s)}[T]^n \quad \text{and}$$
$$R_{1+s}[T]^n \to R_{s(1+s)}[T]^n \to R_{s(1+s)}[T]^n.$$

The map $Q \to I' \to 0$ is got by the property of fibre product.

If $g'\colon Q_s \xrightarrow{\sim} R_s[T]^n$ and $g''\colon Q_{1+s} \xrightarrow{\sim} R_{1+s}[T]^n$ are the obvious isomorphisms, then $(g')_{1+s} \circ (g''^{-1})_s = H = \mathrm{Id} \pmod T$. Hence by ([7], Sect. II, Lemma 2) Q is free of rank n.

Since upper right hand and lower left hand sequences in the diagram are exact, we see that $Q \to I' \to 0$ is exact. Thus I' is generated by n elements and hence $\theta(IR') = I_1 = I'_T$ is generated by n elements. This completes the proof of Theorem 1.2 as indicated before.

Corollary 1.5. *Suppose $R = A[X_1, \ldots, X_n]$ is a polynomial ring in n variables over a commutative noetherian ring A. If I is an ideal in R with*

$$\text{height}\,(I) > \dim A \quad \textit{and} \quad \mu(I/I^2) \geqq \dim(R/I) + 2 \quad \textit{then} \quad \mu(I) = \mu(I/I^2).$$

Proof. As $\text{height}(I) > \dim A$, by a change of variables we can assume I contains a monic polynomial and hence the corollary follows immediately from Theorem 1.2.

Corollary 1.5 settles our question ([4], Chapter III, Remark 2.6) affirmatively. This result was proved by S.M. Bhatwadekar and R.A. Rao ([2], Theorem 1) when A is affine domain.

2. In Laurent polynomial rings

Main result in this section is Theorem 2.2. This theorem is a consequence of Theorem 2.1, which is the Laurent polynomial analogue of Theorem 1.2.

Recall that a Laurent polynomial f in $A[T, T^{-1}]$ is called a doubly monic Laurent polynomial if both the coefficients of the highest degree term and the lowest degree term in f are equal to one.

Theorem 2.1. *Let $R = A[T, T^{-1}]$ be a Laurent polynomial ring over a commutative noetherian ring A in one variable T. Suppose I is an ideal of R, which contains a doubly monic Laurent polynomial. If $\mu(I/I^2) \geqq \dim(R/I) + 2$ then $\mu(I) = \mu(I/I^2)$.*

Proof. Write $I' = I \cap A[T]$ and $J = A \cap I$. Since I contains a doubly monic Laurent polynomial, I' contains special monic.

Suppose $a_1, \ldots, a_n$ form a minimal set of generators of $I \bmod I^2$, where $n = \mu(I/I^2)$. We can assume $a_1, \ldots, a_n$ belongs to I' and with the help of a special monic in I' we can further assume a_1 is a special monic polynomial. We shall see that $a_1, \ldots, a_n$ generates $I' \bmod I'^2$. It is enough to see that for every prime ideal P of $A[T]$, $(I'/I'^2)_P$ is generated by these elements. If T belongs to P then a_1 does not belong to P and hence $(I'/I'^2)_P = 0$. If T does not belong to P, then $(I')_P = (I)_{P_T}$ and hence $(I'/I'^2)_P$ is generated by $a_1, \ldots, a_n$. Hence it follows that $\mu(I/I^2) = \mu(I'/I'^2)$.

Now as both R/I and $A[T]/I'$ are integral extensions of A/J, we have

$$\dim(R/I) = \dim(A/J) = \dim(A[T]/I').$$

Thus
$$\mu(I'/I'^2)=\mu(I/I^2)\geqq\dim(R/I)+2=\dim(A[T]/I')+2.$$

Therefore by an application of Theorem 1.2 we get
$$\mu(I')=\mu(I'/I'^2)=\mu(I/I^2).$$

Hence $\mu(I)\leqq\mu(I/I^2)$. Thus the proof is complete.

Theorem 2.2. *Let $R=A[X_1,\dots,X_n,\ T_1^{\pm 1},\dots,T_r^{\pm 1}]$ with n, $r\geqq 0$ be a Laurent polynomial ring in several variables over a commutative noetherian ring A. Suppose I is an ideal of R with* height $(I)>\dim A$ *and $\mu(I/I^2)\geqq\dim(R/I)+2$. Then $\mu(I)=\mu(I/I^2)$.*

Proof. For $r=0$ it is Corollary 1.5. If $r\geqq 1$ then it is a immediate consequence of Theorem 2.1 and the following lemma.

Lemma 2.3. *Let $R=A[X_1,\dots,X_n,T_1^{\pm 1},\dots,T_r^{\pm 1}]$ with $n\geqq 0$ and $r\geqq 1$ be a Laurent polynomial ring in several variables over a commutative noetherian ring A. Given any ideal I of R with* height$(I)>\dim A$, *there is an A-automorphism θ: $R\to R$ such that, $\theta(I)$ contains a doubly monic Laurent polynomial in T_1.*

For $n=0$ this is a result of Suslin ([10], 7, Lemma 7.1). In [4] we have given an example ([4], Chapter III, Remark 2.5) to show that such a Lemma is not available for polynomial rings i.e. if I is an ideal of $R=A[X_1,\dots,X_n]$ with height$(I)>\dim A$, then I need not contain a special monic via any change of variables.

With this we coclude this section and the proof of Lemma 2.3 will be given in the next Sect. (3).

3. The proof of Lemma 2.3

First we shall set up some notations.

If $R=A[T]$ (resp. $A[T,T^{-1}]$) is a polynomial ring (resp. Laurent polynomial ring) in one variable T over a commutative ring A and I is an an ideal of R then $L_T(I)$ denotes the ideal of A, consisting of coefficients of the highest degree term in T of elements in I. Similarly for an ideal I of $R=A[T,T^{-1}]$, $L_{T^{-1}}(I)$ will denote the ideal of A, consisting of coefficients of the lowest degree term in T of the elements of I. In the case of Laurent polynomial rings $R=A[X_1,\dots,X_n,\ T_1^{\pm 1},\dots,T_r^{\pm 1}]$ in several variables, when we write $L_{X_1}(I)$, $L_{T_1}(I)$ or $L_{T^{-1}}(I)$, we mean R is considered as a polynomial or a Laurent polynomial ring over the rest of the variables and the notations are used in the above sence.

There is a well known result ([1], 4, Lemma 2) which says that if $R=A[T]$ is a polynomial ring and I an ideal of R, then height $(L_T(I))\geqq$ height(I). The following is a easy consequence of this.

Lemma 3.1. *Let* $R=A[T,T^{-1}]$ *be a Laurent polynomial ring over a commutative noetherian ring* A *and* I *an ideal of* R. *Then* $\mathrm{height}(L_T(I))\geqq \mathrm{height}(I)$ *and* $\mathrm{height}(L_{T^{-1}}(I))\geqq \mathrm{height}(I)$.

Proof. It is enough to prove one of the inequalities. We prove the first one. Write $J=I\cap A[T]$. Then $\mathrm{height}(J)=\mathrm{height}(I)$ and $L_T(J)=L_T(I)$. Hence $\mathrm{height}(L_T(I))\geqq \mathrm{height}(I)$ by ([1], 4, Lemma 2).

Now we are ready to prove Lemma 2.3.

Proof of Lemma 2.3. The proof is by induction in two stages. First we prove the Lemma for $r=1$ by induction on n and then use induction on r to complete the proof.

Proof of the Lemma when $r=1$, i.e. $R=A[X_1,\dots,X_n, T, T^{-1}]$. If $n=0$ then $R=A[T,T^{-1}]$ and in view of Lemma 3.1 we have $L_T(I)=L_{T^{-1}}(I)=A$. So we see that I contains an element f which is monic in T and an element g which is monic in T^{-1}. We can combine f and g suitably to get a doubly monic Laurent polynomial in I.

Assume now $r=1$ and $n>0$. We are going to use induction on n to complete the proof in this case. We have $R=A[X_1,\dots,X_n, T, T^{-1}]$. Consider the ideal $L_{X_n}(I)$. We see that $\mathrm{height}(L_{X_n}(I))\geqq \mathrm{height}(I)>\dim A$. Hence by induction hypothesis we may assume (via an A-automorphism of $A[X_1,\dots,X_n, T, T^{-1}]$) that $L_{X_n}(I)$ contains a doubly monic Laurent polynomial f in T. In fact we may assume $f=T^p+g_1T^{p-1}+\dots+g_{p-1}T+1$ for some $p\geqq 1$ and g_i in $A[X_1,\dots,X_n]$, $i=1$ to $p-1$. Let $F(X_n)$ be an element in I with f as the coefficient of its highest degree term. Therefore $F(X_n)=fX_n^q+f_1X_n^{q-1}+\dots+f_q$ for some $q\geqq 1$ and f_j in $A[X_1,\dots,X_{n-1}, T, T^{-1}]$, for $j=1$ to q. Let $s>T$-degree and T^{-1}-degree of f_j for $j=1$ to q. Define θ: $R\to R$ to be the A-automorphism given by $\theta(X_i)=X_i$ for $1\leqq i\leqq n-1$, $\theta(X_n)=X_n+T^s+T^{-s}$ and $\theta(T)=T$. Then $\theta(F(X_n))$ is a doubly monic Laurent polynomial. This completes the proof of the Lemma for $r=1$ and arbitrary $n\geqq 0$.

Proof of the lemma in the general case. Since we have proved the Lemma when $r=1$ and $n\geqq 0$ arbitrary, here we shall apply induction on r to complete the proof.

Assume $r>1$ and $n\geqq 0$.

So we have $R=A[X_1,\dots,X_n, T_1^{\pm1},\dots,T_r^{\pm1}]$. Look at the ideals $L_{T_r}(I)$ and $L_{T_r^{-1}}(I)$ of $A[X_1,\dots,X_n, T_1^{\pm1},\dots,T_{r-1}^{\pm1}]$. Since $\mathrm{height}(I)>\dim A$, by Lemma 3.1 both $L_{T_r}(I)$ and $L_{T_r^{-1}}(I)$ have heights strictly greater than $\dim A$ and hence $\mathrm{height}(L_{T_r}(I)\cap L_{T_r^{-1}}(I))>\dim A$. By induction hypothesis (via an A-automorphism of $A[X_1,\dots,X_n, T_1^{\pm1},\dots,T_{r-1}^{\pm1}]$), $L_{T_r}(I)\cap L_{T_r^{-1}}(I)$ contains a Laurent polynomial f which is doubly monic in T_1. We may write $f=T_1^p+g_1T_1^{p-1}+\dots+g_{p-1}T_1+1$ for some $p\geqq 1$ and g_i in $A[X_1,\dots,X_n, T_2^{\pm1},\dots,T_{r-1}^{\pm1}]$. So we can find F and G in I such that $F(T_r)=fT_r^q+f_1T_r^{q-1}+\dots+f_{q-1}T_r+f_q$ and $G(T_r)=f+h_1T_r+\dots+h_uT_r^u$ for some integers q, $u\geqq 0$ and f_i, h_j in $A[X_1,\dots,X_n, T_1^{\pm1},\dots,T_{r-1}^{\pm1}]$ for $i=1$ to q and $j=1$ to u.

Let $s > T_1$-degree of f_i and T_1^{-1}-degree of h_j, for $i=1$ to q and $j=1$ to u. Define an A-automorphism θ: $R \to R$ as follows,

$$\begin{aligned} \theta(X_i) &= X_i && \text{for } 1 \leqq i \leqq n \\ \theta(T_i) &= T_i && \text{for } 1 \leqq i \leqq r-1 \\ \theta(T_r) &= T_r T_1^s && \text{for } i = r. \end{aligned}$$

Then $T_r^{-q}\theta(F)$ is monic in T_1 and $\theta(G)$ is monic in T_1^{-1} over the rest of the variables. Hence a suitable linear combination H of $T_r^{-q}\theta(F)$ and $\theta(G)$ can be found which is doubly monic in T_1. As H is an element of $\theta(I)$, the proof of Lemma 2.3 is complete.

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Note added in proof

R.A. Rao has shown how Remark 1.3 can be used to prove that if $(f_1, \ldots, f_n)$ is a unimodular row in a polynomial ring $A[X]$ with f_1 a monic polynomial and $n \geqq 3$ then there is an elementary matrix u which takes $(f_1, \ldots, f_n)$ to $(1, 0, \ldots, 0)$.