Some Examples and Cosntructions

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1 Introduction

I will talk on the construction of N. Mohan Kumar in his paper entitled "Stably free modules" ([MK1]). This paper is widely known for the examples of stably free projective modules that are not free.

In fact, such examples of stably free projective modules in this paper is (only) an application of the main construction in this paper.

We will discuss two further applications and/or aspects of the main construction.

This is part of an on going work with S. M. Bhatwadekar and Mrinal Kanti Das.

Part of the goal of this talk is to produce a smooth three fold $X = Spec(A)$ and a projective $A-$module P with $rank(P) = 3$, such that $[P] - [A^3] \neq 0$ in the Grothendieck group $K_0(A)$ and all the Chern classes $C^r(P) = 0$ for all $r \geq 1$.

In contrast, recall the theorem of Mohan Kumar and Murthy [MKM]:

Theorem 1.1 ([MKM]) Let A be a smooth affine ring over an algebraically closed field and $\dim(A) = 3$. Suppose P is a projective $A-$module with any $rank(P) = r \geq 1$. Then

$$[P] - [A^r] = 0 \Longleftrightarrow C^k(P) = 0 \quad for \quad all \quad k \geq 1.$$

In other words, a projective $A-$module is stably free if and only if all the Chern classes of P are zero.

2 Mohan Kumar Varieties

The following is the constructions of Mohan Kumar ([MK1]).

Construction 2.1 [MK1, Mohan Kumar] Let k be a field and p be a prime number. Fix a polynomial $f(x)$ of degree p over k such that $f(0) = a \in k^*$. This polynomial $f(x)$ will be called the **seed polynomial.** Let

$$t_r = 1 + p + \cdots + p^{r-1} = \frac{p^r - 1}{p - 1}.$$

1. Let $F(x_0, x_1) = F_1(x_0, x_1) = x_1^p f(x_0/x_1)$.
2. Inductively define
$$F_n = F(F_{n-1}(x_0, \ldots, x_{n-1}), a^{t_{n-1}} x_n^{p^{n-1}}).$$
3. We will work with (seed) polynomials $f(x)$ so that F_n is irreducible (for appropriate n).

4. Let $S_n = V(F_n) \subseteq \mathbb{P}_k^n$ denotes the closed subset of $\mathbb{P}_k^n$ defined by $F_n = 0$. That means

$$S_n = Proj\left(\frac{k[x_0, x_1, x_2, \ldots, x_n]}{F_n}\right).$$

These varieties S_n will be called a **Mohan Kumar (projective) varieties.** Note that these are singular varieties.

5. Let $X_n = \mathbb{P}_k^n \setminus S_n$. Write $X_n = Spec(A_n)$. Then

$$A_n = k[x_0, x_1, x_2, \ldots, x_n]_{(F_n)},$$

the homogneous localization. These varieties X_n will be called **Mohan Kumar (affine) varieties.** Note that these are non-singular affine varieties.

6. Often, out seed polynomial is $f(x) = x^p + a$ where $k = k_0(a)$ and a is trnacendental over a field k_0.

Among other things, Mohan Kumar ([MK1]) did the following computations regarding the Chow groups.

Theorem 2.2 ([MK1, Mohan Kumar]) Let p be a prime number and let rest of the notations be as above (2.1). Then,

1. The Chow group $CH^1(X_n)$ of codimension one cycles is given by
$$CH^1(X_n) = \mathbb{Z}/(p^n).$$
Proof. Note $degree(F_n) = p^n$. The rest of the proof follows from the exact sequence
$$CH^0(S_n) = \mathbb{Z} \to CH^1(\mathbb{P}^n_k) = \mathbb{Z} \to CH^n(X_3) \to 0.$$

2. Also, the Chow group $CH^n(X_n)$ of codimension n cycles (i.e. zero cycles) is given by
$$CH^n(X_n) = \mathbb{Z}/(p).$$
Proof. This follows from the exact sequence
$$CH^{n-1}(S_n) \to CH^n(\mathbb{P}^n_k) = \mathbb{Z} \to CH^n(X_n) \to 0$$
and the fact that $image(CH^{n-1}(S_n)) = p\mathbb{Z}$. This follows by observing that degree of any closed point $\wp \in S_n$ is divisible by p and the closed point $\wp_0 = (F_1, x_2, x_3, \dots, x_n) \in S_n$ has degree p.

3. **Problems:** The complete description of the total Chow group $CH(X_n)$ is not known. Also the description of the Grothendieck group $K_0(X_n)$ is not known.

2.1 Computation of Grothendieck $K-$Groups

Our question is $K-$theoretic. So, we wish to gather information regarding $K-$groups.

Theorem 2.3 Let $k = k_0(a)$ where a is trnacendental over a field k_0 and let $f(x) = x^2 + a$ be the seed polynomial. (*We consider Mohan Kumar varieties with* $p = 2, n = 3$.) Consider the exact sequence

$$G_0(S_3)) \xrightarrow{i} K_0(\mathbb{P}^3) \to K_0(A_3) \to 0.$$

We have

$$i(G_0(S_3)) \subseteq 2K_0(\mathbb{P}^3).$$

Proof. We will write $\mathcal{O} = \mathcal{O}_{\mathbb{P}^3}$. Recall

$$K_0(\mathbb{P}^3) = \mathbb{Z}[T]/(T-1)^4 = \sum_{k=0}^{3} \mathbb{Z}\eta^k \quad where \quad \eta = [\mathcal{O}(-1)].$$

Given any point $x \in S_3$, let $\mathcal{P}$ denote the ideal sheaf of x. We will prove that

$$\left[\frac{\mathcal{O}}{\mathcal{P}}\right] \in 2K_0(\mathbb{P}^3).$$

We will write $R = k[x_0, x_1, x_2, x_3]$.

Step I : The generic point in S_3. Let $x \in S_3$ be the generic point of S_3. Consider the exact sequence

$$0 \to R(-2^3) \xrightarrow{F_3} R \to \frac{R}{F_3} \to 0.$$

Looking at the shief, we have the exact sequence:

$$0 \to \mathcal{O}(-2^3) \to \mathcal{O} \to \frac{\mathcal{O}}{\mathcal{P}} \to 0.$$

Therefore,

$$\left[\frac{\mathcal{O}}{\mathcal{P}}\right] = [\mathcal{O}] - [\mathcal{O}(-2^3)] = (1 - \eta^{2^3}) \overset{mod\ 2}{\equiv} (1 - \eta)^8 \overset{mod\ 2}{\equiv} 0$$

Step II : Closed points in S_3. Let $x \in S_3$ be a closed point and $m \in \mathbb{P}^3$ be the corresponding homogeneous-prime ideal. So, $F_3 \in m$ and $height(m) = 3$. By theorem 2.2, item 2, we have $cycle(\frac{\mathcal{O}}{\mathcal{P}}) = 2d \in CH^3(\mathbb{P}^3)$ for some integer d. Now consider the maps

$$CH^3(\mathbb{P}^3) \xrightarrow{\beta} F^3K_0(\mathbb{P}^3).$$

Write

$$\mathcal{I} = ideal - Sheaf - generated(F_1^d, x_2, x_3).$$

Now

$$cycle\left(\frac{\mathcal{O}}{\mathcal{P}}\right) = cycle\left(\frac{\mathcal{O}}{\mathcal{I}}\right) \quad in \quad CH^3(\mathbb{P}^3).$$

Therefore, looking at the image under beta (Fulton [F, page 285]), we have

$$\left[\frac{\mathcal{O}}{\mathcal{P}}\right] = \left[\frac{\mathcal{O}}{\mathcal{I}}\right] \quad in \quad F^3K_0(\mathbb{P}^3).$$

There is a surjective map

$$R(-2d) \oplus R(-1)^2 \to (F_1^d, x_2, x_3) \to 0$$

that induces a surjective map

$$\mathcal{O}(-2d) \oplus \mathcal{O}(-1)^2 \to \mathcal{I} \to 0.$$

The Koszul complex gives (see 4.1):

$$\left[\frac{\mathcal{O}}{\mathcal{I}}\right] = (\mathcal{O}(-2d) - 1)(\mathcal{O}(-1) - 1)^2$$

$$= (\eta^{2d} - 1)(\eta - 1)^2 \stackrel{mod\ 2}{\equiv} (\eta^d - 1)^2(\eta - 1)^2 = 0$$

because $(\eta - 1)^4 = 0$. Therefore,

$$\left[\frac{\mathcal{O}}{\mathcal{P}}\right] = \left[\frac{\mathcal{O}}{\mathcal{I}}\right] \quad is \quad in \quad 2K_0(\mathbb{P}^3).$$

Step III: $codim = 2$**:** Let $x \in S_3$ be a codimension two point. We think of $x = \wp$ as a homogeneous prime ideal of height 2 and let $\mathcal{P} = IdealSheaf(\wp)$.
Case $x_3 \in \wp$**:** Suppose $x_3 \in \wp$. Then $\wp = (F_2, x_3)$ The surjective map

$$R(-4) \oplus R(-1) \to \wp$$

induces surjective map

$$\mathcal{O}(-4) \oplus \mathcal{O}(-1) \to \mathcal{P}.$$

The Koszul complex gives (see 4.1):

$$\left[\frac{\mathcal{O}}{\mathcal{P}}\right] = (1 - \eta^4)(1 - \eta) \stackrel{mod\ 2}{\equiv} (1 - \eta)^5 \stackrel{mod\ 2}{\equiv} 0$$

because $(1 - \eta)^4 = 0$.

Case $x_3 \notin \wp$: Write $y_i = x_i/x_3$. $f_3 = F_3(y_0, y_1, y_2, 1) \in \wp_{(x_3)}$. By lemma 2.4,

$$\frac{k[y_0, y_1, y_2]}{(f_3)} \quad is \quad an \quad UFD.$$

It follows that $\wp_{(x_3)} = (f_3, h)$ for some $h \in k[y_0, y_1, y_2]$. Let $h = H(x_0, x_1, x_2, x_3)/x_3^k$ for some homogneous $H \in \wp$ so that $H(x_0, x_1, x_2, 0) \neq 0$. **Note H is not a multiple of F, otherwise $\wp_{(x_3)} = (f_3)$. This is impossible. Therefore F, H is a regular sequence.**

Let

$$I = (F_3, H) \qquad and \quad \mathcal{I} = IdealSheaf(I).$$

Let $degree(H) = d$. Then, F_3, H induces a surjective map

$$\mathcal{O}(-8) \oplus \mathcal{O}(-d) \to \mathcal{I} \to 0.$$

The Koszul complex gives (see 4.1):

$$\left[\frac{\mathcal{O}}{\mathcal{I}}\right] = (1-\eta^8)(1-\eta^d) \overset{mod\ 2}{\equiv} (1-\eta)^8(1-\eta^d) \overset{mod\ 2}{\equiv} 0$$

because $(1-\eta)^4 = 0$. Therefore

$$2 \quad divides \quad \left[\frac{\mathcal{O}}{\mathcal{I}}\right].$$

We have the exact sequence

$$0 \to \frac{\mathcal{P}}{\mathcal{I}} \to \frac{\mathcal{O}}{\mathcal{I}} \to \frac{\mathcal{O}}{\mathcal{P}} \to 0$$

Also note the following:

1. $$\frac{\mathcal{P}}{\mathcal{I}} = Sheaf(\frac{\wp}{I})$$

2. $(I, x_3) \subseteq \sqrt{ann(\frac{\wp}{I})}$. **Proof.** We have $\wp_{(x_3)} = I_{(x_3)}$. So, $(\frac{\wp}{I})_{(x_3)} = 0$. Therefore $x_3^r(\frac{\wp}{I}) = 0$ for some r.

3. It is clear that $\wp_0 = (F_2, x_3)$ is the ONLY height-two prime ideal that MAY contain (I, x_3).

4. There are graded submodules

   $$M_0 = 0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_N = \frac{\wp}{I}$$

   with
   $$\frac{M_i}{M_{i+1}} \equiv \frac{k[x_0, x_1, x_2, x_3]}{\wp_i}$$
   where $(I, x_3) \subseteq \wp_i$ are homogeneous primes for $i = 1, \ldots, N$

5. Let
   $$\mathcal{P}_i = IdealSheaf(\wp_i).$$

6. By looking at the corresponding sheafs we have

$$\left[\frac{\mathcal{P}}{\mathcal{I}}\right] = \left[Sheaf(\frac{\wp}{I})\right] = [Sheaf(M_N)] = \sum_{i=1}^{N} \left[\frac{\mathcal{O}}{\mathcal{P}_i}\right].$$

7. For $i = 1, \ldots, N$ if $\wp_i \neq \wp_0 = (F_2, x_3)$ then $height(\wp_i) \geq 3$.

8. By downward induction or because $\wp_i = (F_2, x_3)$ we have

$$2 \qquad devides \qquad \left[\frac{\mathcal{O}}{\mathcal{P}_i}\right] \quad for \quad i = 1, \ldots, N.$$

Therefore

$$\left[\frac{\mathcal{P}}{\mathcal{I}}\right] = \sum_{i=1}^{N} \left[\frac{\mathcal{O}}{\mathcal{P}_i}\right] \quad divisible \quad by \quad 2.$$

9. The exact sequence

$$0 \longrightarrow \frac{\mathcal{P}}{\mathcal{I}} \longrightarrow \frac{\mathcal{O}}{\mathcal{I}} \longrightarrow \frac{\mathcal{O}}{\mathcal{P}} \longrightarrow 0$$

gives

$$2 \quad divides \quad \left[\frac{\mathcal{O}}{\mathcal{P}}\right] = \left[\frac{\mathcal{O}}{\mathcal{I}}\right] - \left[\frac{\mathcal{P}}{\mathcal{I}}\right].$$

This completes the proof of the theorem.

Lemma 2.4 Let k_0 be a field and $k = k_0(t)$ and t is trancendental over a field k_0. Let y_0, y_1, y_2 be variables and $A = k[y_0, y_1, y_2]$ and

$$f_3 = F_3(y_0, y_1, y_2, 1) = (((y_0^2 + ty_1^2)^2 + t^3y_2^4)^2 + t^7.$$

Then $A/(f_3)$ is an UFD.

2.2 The Examples

As promised in the introduction, we produce the following example from the computaitons in Theorem 2.3.

Corollary 2.5 Suppose $k = k_0(a)$ where a is trnacendental over a field k_0 and let $f(x) = x^2 + a$ be the seed polynomial. Let $X_3 = Spec(A_3)$ be the Mohan Kumar affine three fold. Let m be a $k-$rational maximal ideal in A_3 and $x = [\frac{A_3}{m}] \in K_0(A_3)$.

1. Then $x \neq 0$.
2. There is a projective module P of rank three such that $x = [P] - [A_3^3]$ then $C^r(P) = 0$ for all $r \geq 1$.

Proof. We have the exact sequence

$$G_0(F_3 = 0)) \xrightarrow{\psi} K_0(\mathbb{P}^3) \to K_0(X_3) \to 0$$

Let $y \in K_0(\mathbb{P}^3)$ is given by a rational point. So $\psi(y) = x$. Write $R = k[x_0, x_1, x_2, x_3]$. Note that the exact sequence

$$R(-1)^3 \to (x_0, x_1, x_2) \quad gives \quad y = (1 - \eta)^3.$$

Since y is not a multiple of 2, we have $x \neq 0$.

Now we prove part two. Since $x = [P] - [A_3^3]$ is supported in codimention three, $C^1(x) = C^2(x) = 0$. Also, by Riemann-Roch, $C^3(x) = 2[cycle(x)] = 0$ because $CH^3(A_3) = \mathbb{Z}/(2)$.

By taking a common denominator, the following is a consequence of the above example 2.5.

Corollary 2.6 There is an affine smooth algebra A over $\mathbb{C}$ with $\dim(A) = 4$ and a projective $A-$module P with $rank(P) = 3$ such that $[P] - [A^3] \neq 0$ and $C^r(P) = 0$ for all $r \geq 1$.

This shows that the theorem of Mohan Kumar and Murthy ([MKM]) is not valid for higher dimension.

Following is a consequence of the UFD lemma.

Corollary 2.7 Let $k = k_0(a)$ where a is trnacendental over a field k_0 amd let $f(x) = x^2 + a$ be the seed polynomial. We have

$$CH^2(X_3) = \mathbb{Z}/(4).$$

So, in this case, the total Chow group is given by

$$CH(X_3) = \mathbb{Z} \oplus \mathbb{Z}/(8) \oplus \mathbb{Z}/(4) \oplus \mathbb{Z}/(2)$$

Proof. We have notations $S_n = (F_n = 0) \subseteq \mathbb{P}^n$ and $X_n = \mathbb{P}^n \setminus S_n$.

Concider the diagram:

$$\begin{array}{ccccccc}
CH^0(F_2 = 0, x_3 = 0) = \mathbb{Z} & \xrightarrow{\psi_0} & CH^1(S_3) & \longrightarrow & CH^1(S_3 \cap (x_3 \neq 0)) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \beta_1} & & \downarrow & & \\
CH^1(\mathbb{P}^2) = \mathbb{Z} & \xrightarrow{\psi_1} & CH^2(\mathbb{P}^3) = \mathbb{Z} & \longrightarrow & CH^2(\mathbb{A}^3) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
CH^1(X_2) & \longrightarrow & CH^2(X_3) & \longrightarrow & CH^2(X_3 \setminus X_2) & \longrightarrow & 0
\end{array}$$

Since $\frac{k[y_0,y_1,y_2]}{(F_3(y_0,y_1,y_2,1)}$ is a UFD, $CH^1(S_3 \cap (x_3 \neq 0)) = 0$. (Hence all of the third column is zero, but we do not need it.) Now it follows that ψ_0 is surjective. So, Let $\zeta = (F_2, x_3)$ denote the generator of $CH^0(F_2 = 0, x_3 = 0)$. Then $\beta_1(CH^1(S_3)) = \beta_1(\zeta)\mathbb{Z} = 4\mathbb{Z}$ So, $CH^2(X_3) = \mathbb{Z}/(4)$.

Theorem 2.8 Let $k = k_0(a)$ where a is trancendental over a field k_0. Let $f(x) = x^2 + a$ be the seed polynomial. Let X_3 be a Mohan Kumar affine three fold ovet k. Then the Grothendieck gropup $K_0(X_3)$ is also completely determined.

3 Over the reals $\mathbb{R}$

For our second application, we consider Mohan Kumar varieties over the field of reals $\mathbb{R}$. Our seed polynomial is $f(x) = x^2 + 1 \in \mathbb{R}[x]$.

Proposition 3.1 Let $f(x) = x^2 + 1 \in \mathbb{R}[x]$. Let $F_1 = F(x_0, x_1) = x_0^2 + x_1^2$. Define

$$F_n(x_0, x_1, \ldots, x_n) = F(F_{n-1}, x_n) = F_{n-1}^2 + x_n^{2^n}.$$

Then F_n is irreducible in $\mathbb{R}[x_0, x_1, \ldots, x_n]$.

All the computations of Mohan Kumar ([MK1]) goes through. We rewrite theorem 2.2 in this context.

Theorem 3.2 Let $f(x) = x^2 + 1 \in \mathbb{R}[x]$ be the seed polynomial. Then,

1. The Mohan Kumar affine variety $X_n = Spec(A_n) = \mathbb{P}^n_{\mathbb{R}} \setminus (F_n = 0)$ is a smooth real affine variety of dimension n.

2. We have $CH^1(X_n) = \mathbb{Z}/(2^n)$.

3. Also, $CH^n(X_n) = \mathbb{Z}/(2)$.

4. Recall ([H]), the cannonical bundle

$$K_{\mathbb{P}^n} = \wedge^n \Omega_{\mathbb{P}^n} = \mathcal{O}(-n-1).$$

So,

$$K_{X_n} = -n-1 \quad in \quad Pic(X_n) = CH^1(X_n) = \mathbb{Z}/(2^n).$$

We compute the Euler class group of the Mohan Kumar affine variety over reals as follows:

Theorem 3.3 Suppose $L \in Pic(X_n) = \mathbb{Z}/(2^n)$ is a line bundle on X_n. If $L + n + 1$ is even then

$$E(X_n, L) = \mathbb{Z}$$

and if $L + n + 1$ is odd then

$$E(X_n, L) = \mathbb{Z}/(2)$$

Proof.

1. **Notation:** For any affine variety $X = Spec(A)$ over $\mathbb{R}$, let

   $$X(\mathbb{R}) = Spec(S^{-1}A) \quad and \quad \mathbb{R}(X) = S^{-1}A,$$

   where S is the set all all $f \in A$ that does not vanish at any real point.

2. $X_n(\mathbb{R}) = P^n(\mathbb{R})$. So they have one connected component, which is compact.

3. $Pic(X_n) = \mathbb{Z}/(2^n)$. In fact,

   $$Pic(X_n(\mathbb{R})) = \mathbb{Z}/(2).$$

4. The cannonical bundle is $K = -(n+1)$.

5. We have the follwoing diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E^{\mathbb{C}}(L) & \longrightarrow & E(X_n, L) & \longrightarrow & E(\mathbb{R}(X_n), L) & \longrightarrow & 0 \\ & & \wr\big\downarrow\varphi & & \big\downarrow\Theta & & \big\downarrow & & \\ 0 & \longrightarrow & K(\mathbb{C}) & \longrightarrow & CH^n(X_n) & \longrightarrow & CH^n(\mathbb{R}(X_n)) & \longrightarrow & 0 \end{array}$$

6. Since $CH^n(X_n) = \mathbb{Z}/(2)$ and $CH^n(\mathbb{R}(X_n)) = \mathbb{Z}/(2)$ we have
$$E(X_n, L) = E(\mathbb{R}(X_n), L)$$
Therefore, if $L + n + 1$ is even then
$$E(X_n, L) = E(\mathbb{R}(X_n), L) = \mathbb{Z}$$
and if $L + n + 1$ is odd then
$$E(X_n, L) = E(\mathbb{R}(X_n), L) = \mathbb{Z}/(2).$$

4 Appendix

4.1 Koszul Complex Lemma

Exercise 4.1 *Let X be noetherian scheme and $\mathcal{I}$ be local complete intersection sheaf of ideal of codimension r. Suppose $L_1, \ldots, L_r$ are line bundles and*

$$L_1 \oplus L_2 \oplus \cdots \oplus L_r \to \mathcal{I}$$

be a surjective map. Then

$$\left[\frac{\mathcal{O}}{\mathcal{I}}\right] = (1 - [L_1])(1 - [L_2]) \cdots (1 - [L_r]) \quad in \quad K_0(X)$$

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